3. Blow-up in global geometric situation

group scheme structure

$\downarrow$

differential form

rigid geom $/K$ schemes $/S = \operatorname{Spec} \mathcal{O}_K$

$$
\begin{tikzcd}
G \subset D^n \arrow[d, "f"'] & \supset & f^{-1}(D(0,r)) \arrow[d] \\
0 \in D^n & \supset & D(0,r)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_L \arrow[r] \arrow[d] & Q \arrow[d] & Q^{(r)}_{S'} \arrow[l] \arrow[d] \\
S \arrow[r] & P \arrow[d] & P^{(r)}_{S'} \arrow[l] \\
 & S &
\end{tikzcd}
$$

$X * S$

X smooth sch / k perfect of char $p > 0$

$\cup$

D divisor w. S.N.C

$\shortparallel$

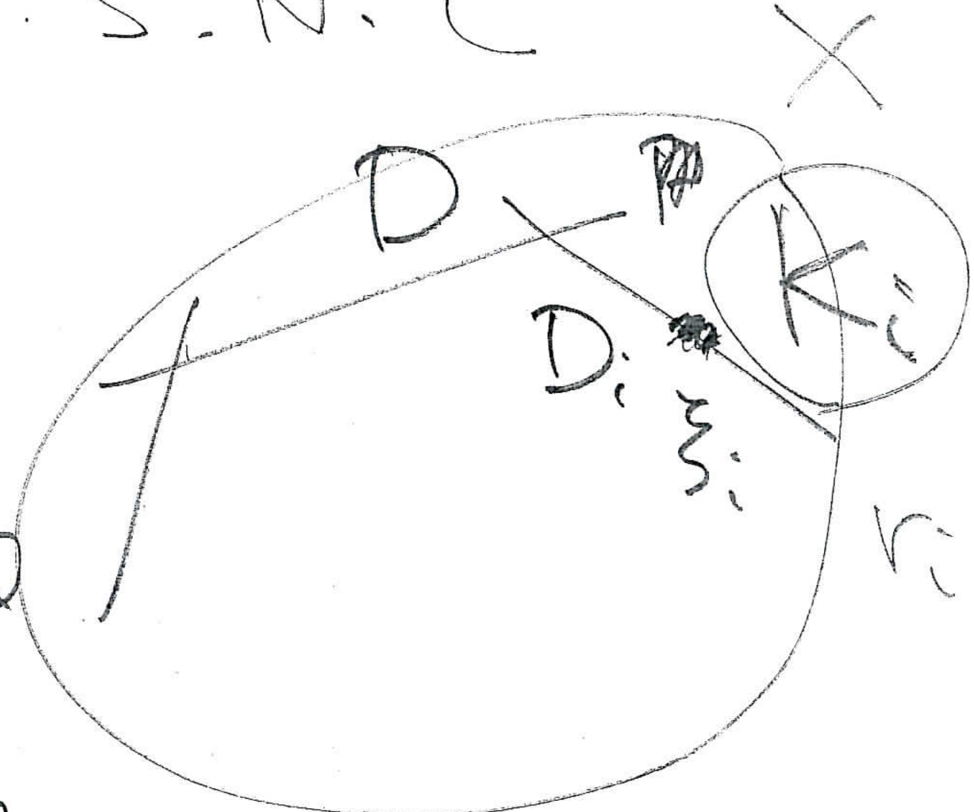
$\cup D_i$

$R = \sum r_i D_i$

$r_i \geq 0$ rational

Simplifying ass'n

$r_i > 0$: integer.



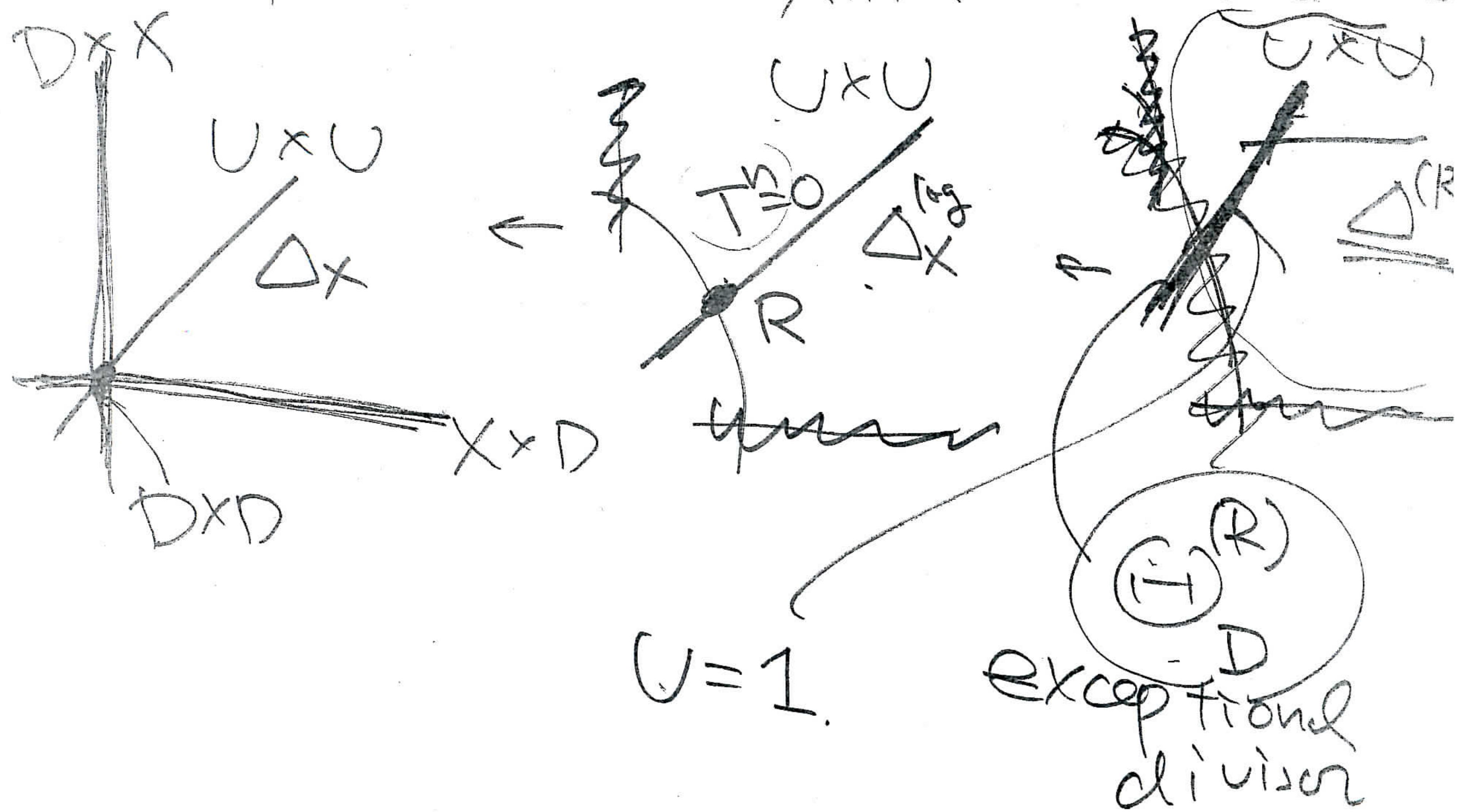
$$\begin{tikzcd} X \subseteq X\times X & X * X \arrow[l] & (X*X)^{(R)} \arrow[l] \\ X \arrow[u, "\Delta_X"] \arrow[ur, "\Delta_X^{\log}"] \arrow[urr, "\Delta_X^{(R)}"'] & & \\ R \arrow[u, phantom, "\cup"] & & \end{tikzcd}$$

effective Cartier divisor R

blow-up at R & remove the proper transform of $(X*X)\times D \xrightarrow{pr_2} X$

$X \times X \supset U \times U$

$X * X$. $U \times U \subset (X * X)^{(R)}$



$$\begin{tikzcd} Y \arrow[d, "\text{finite}"'] & V \arrow[l, hook'] \arrow[d, "\text{finite étale Galois.}\quad G"] \\ X & U = X \setminus D \arrow[l, hook'] \end{tikzcd}$$

normalization of X in V

$\Delta G \curvearrowright Y$

normalization

$(G \times G \supset \Delta G \curvearrowright W$

$$\begin{tikzcd}
Y/G = X \arrow[r] \arrow[d, Rightarrow] & Z \arrow[d] & W = V \times V/\Delta G \arrow[l, hook'] \arrow[d, "\text{finite étale}"] \\
X \arrow[r, "\Delta^{(R)}"] \arrow[ur, bend left] & (X * X)^{(R)} & U \times U \arrow[l, hook']
\end{tikzcd}$$

$$
\begin{tikzcd}
\Theta_D^{(R)} \arrow[r, phantom, "\subset"] \arrow[d] & (X * X)^{(R)} \arrow[d] \\
D \arrow[r, hook] & X
\end{tikzcd}
$$

group str

vector b'dle associated to

$\otimes \mathcal{O}_X(R)$

$$\Theta_D^{(R)} = \mathbb{V}\big(\underbrace{\Omega^1_X(\log D)(R)}_{\mathcal{O}_X} \otimes \mathcal{O}_D\big)$$

locally free $\mathcal{O}_D$-module

$$= \mathrm{Spec}\, S^\bullet_{\mathcal{O}_D}\big(\text{———}\big)$$

Example

$$X = \mathbb{A}^1 = \operatorname{Spec} k[T] \supset D = (0) \qquad p^{th} \qquad r = n$$

$$\cup$$

$$U = \mathbb{G}_m \qquad V \longrightarrow U \qquad t^p - t = \frac{1}{T^n}$$

$$R = nD \qquad G = \operatorname{Gal}(V/U) = \mathbb{F}_p$$

$$(X * X)^{(R)} \qquad X * X = \operatorname{Spec} k[S, T, (\tfrac{S}{T})^{\pm 1}]$$

$$\|$$

$$\operatorname{Spec} k[U^{\pm 1}, T, \frac{U-1}{T^n}] = \operatorname{Spec} k[U^{\pm 1}, T]$$

$$S = U \cdot T.$$

$$(X * X)^{(R)} = \operatorname{Spf} k[U^{\pm 1}, T, V] \,/\, (U - (1 + VT^n))$$

$$U$$

$$U \times U = \operatorname{Spf} k[T^{\pm 1}, S^{\pm 1}]$$

$$W \to U \times U \qquad t^p - t = \frac{1}{S^n} - \frac{1}{T^n}$$

$$\frac{1}{S^n} - \frac{1}{T^n} = \frac{1}{T^n}\left(U^{-n} - 1\right) = \frac{1}{T^n}\left((1 + VT^n)^{-n} - 1\right)$$

$$= \frac{1}{T^n}\big(\cancel{1} - nVT^n + \underbrace{\cdots}_{} - \cancel{1}\big)$$

↑ has no pole on $(X * X)^{(R)}$

We have killed ramification by blow up.

We want to see differential form

$$\begin{tikzcd} \mathbb{A}^1 \cong \Theta^{(R)}_D \arrow[r, phantom, "\subset"] \arrow[d] \arrow[dr, phantom, "\square"] & (X * X)^{(R)} = \mathrm{Spec}\, k[U^{\pm 1}, T, V] \arrow[d] \\ D \arrow[r, phantom, "\subset"] & X = \mathrm{Spec}\, k[T] \end{tikzcd}$$

$T = 0$

$\mathrm{Spec}\, k[V]$

$$\begin{tikzcd}
Z \times \Theta^{(R)}_D \arrow[r, phantom, "\subset"] \arrow[d, "(X*X)^{(R)}"'] & Z \arrow[d] \\
\Theta^{(R)}_D \arrow[r, phantom, "\subset"] \arrow[r, phantom, "T=0"', shift right=3] & (X*X)^{(R)}
\end{tikzcd}$$

$t^P - t = -nV + \cdots$

divisible by T^4

$\Theta^{(R)}_D = \operatorname{Spec} k[V]$

$t^P - t = -nV$

$$V = \frac{U - 1}{T^n} = \frac{1}{T^n} \frac{S - T}{T}$$

$$T \otimes 1 \qquad 1 \otimes T$$

$$dT$$

$$= \frac{1}{T^n} \cdot d\log T$$

$$t^p - t = -n \cdot \left(\frac{1}{T^n} d\log T \right)$$

groupoid structure

$$\begin{tikzcd}
G \arrow[d] & & & \\
(X \times X) \arrow[dr, "pr_2"'] & \times & (X \times X) \arrow[dl, "pr_1"] \arrow[r, "pr_{13}"] & X \times X \\
& X & &
\end{tikzcd}$$

$$\|$$

$$X \times X \times X$$

$$\begin{tikzcd}
Z \times_X Z \arrow[r] & Z \\
(X * X)^{(R)} \times_X (X * X)^{(R)} \arrow[r, "\mu"] & (X * X)^{(R)} \\
\cup \arrow[r, phantom, "\circlearrowright"] & \cup \\
(\Theta)^{(R)}_D \times_D (\Theta)^{(R)}_D \arrow[r, "+"] & (\Theta)^{(R)}_D \\
W \times_U W \arrow[r] \arrow[d] & W = V\times V/ \arrow[d] \\
(U \times U) \times_U (U \times U) \arrow[r, "pr_{13}"] & U \times U
\end{tikzcd}$$

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$$\begin{array}{c} V \\ \perp \\ U \end{array} \qquad G.$$

$D = \cup D_i$. K_i local

$G \supset G_i$ decomposition gp

G_i^{ur} $\quad r_i$ the last jump

$G_i^{s} = 1 \iff s > r_i$

$R = \sum \underline{\underline{r_i}} D_i$

$$\begin{array}{ccc} Z_0 & \subset & Z \\ & & \downarrow \\ & & (X * X)^{(R)} \end{array}$$

maximal open subscheme étale over $(X * X)^{(R)}$

$$\begin{array}{ccc} Z_0 \times_X Z_0 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ (X * X)^{(R)} \times_X (X * X)^{(R)} & \xrightarrow{M} & (X * X)^{(R)} \end{array}$$

Assume

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[ur] \arrow[r, "\Delta^{(R)}"] & (X \times X)^{(R)} \end{tikzcd}$$

the image of $X \subset Z$

is inside Z_0

i.e Z is étale on a nbhd

of X.

b.c D hom of gp sch

$$\begin{tikzcd}
Z_{0,D} \times_D Z_{0,D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Z_{0,D} \arrow[d, "\text{gp}"] \\
\Theta_D^{(R)} \times_D \Theta_D^{(R)} \arrow[r, "+"] & \Theta_D^{(R)}
\end{tikzcd}$$

$Z_{0,D}$ is a smooth group schm

étale over $\Theta_D^{(R)}$

étale isogeny over a vector b'dle over $\Theta_D^{(R)}$

$$D = \bigcup D_i \qquad D_i \ni \underline{\xi_i}$$

$$\underline{Z_{0,\xi_i}} \xrightarrow{\text{étale isogeny}} \Theta^{(R)}_{\xi_i}$$

extension of $\Theta^{(R)}_{\xi_i}$ by $\mathbb{F}_p$-v.sp.

vector space over $F_i = \kappa(\xi_i)$

function field of D_i

$$\mathrm{Ker}\left(\longrightarrow\right) \simeq G^{r_i} \leftarrow \text{elementary } p\text{-gp}$$

$$\mathrm{Ext}^1\left(\Theta^{(R)}_{\xi_i}, \mathbb{F}_p\right) = \mathrm{Hom}\left(\Theta^{(R)}_{\xi_i}, \mathbb{G}_a\right)$$

$$\Vert$$

$$\{\text{linear form on } \Theta^{(R)}_{\xi_i}$$

$$\mathrm{Ext}^1_\tau = \mathrm{Hom}\left(\mathfrak{m}_{K_i}^{r_i}/\mathfrak{m}_{K_i}^{r_i+1},\ \Omega^1_X(\log D) \otimes_{\mathcal{O}_{X,\xi_i}} F_i\right)$$