2-18
3-0

L/K finite Galois ext'n. $G = \mathrm{Gal}(L/K)$

geometric origin

Assume X smooth / k perfect.

$D \subset X$ smooth irred div.

$\xi \in D$ gen. pt $K = \mathrm{Frac}(\hat{\mathcal{O}}_{X,\xi})$ $\mathcal{O}_K$

$$
\begin{tikzcd}
E \arrow[d] & Y \arrow[l, phantom, "\supset"] \arrow[d] & V \arrow[l] \arrow[d, "\text{finite étale Galois}"] \arrow[dl, phantom, "\square"] & \mathrm{Spec}\, L \arrow[l] \arrow[d] \\
D & X \arrow[l, phantom, "\supset"] & U = X \setminus D \arrow[l] & \mathrm{Spec}\, K \arrow[l]
\end{tikzcd}
$$

$G = \mathrm{Gal}(V/U)$

$$\begin{tikzcd}
 & & \mathrm{Spec}\,O_K \arrow[d, "\Delta^{\log}_X"] & X \arrow[d, "\Delta_X"] \\
U_K \arrow[r] \arrow[d] & P = X \times S \arrow[r] \arrow[d, bend right] & X \times X \arrow[r] \arrow[d, "pr_2"] \arrow[dr, phantom, ""] & X \times X \arrow[dl, "pr_2"] \\
\mathrm{Spec}\,K \arrow[r] & \mathrm{Spec}\,O_K = S \arrow[r] \arrow[u, bend left] & X \arrow[u, bend left, "\Delta^{\log}_X"] & \\
 & & X \supset U &
\end{tikzcd}$$

$$\begin{tikzcd}
U \times_k \mathrm{Spec}\,K = U_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d] \\
\mathrm{Spec}\,K \arrow[r] & U
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Spm}\, \mathcal{O}_L = T \arrow[r] \arrow[d, "j"] & Q \arrow[d, "\text{normalization}"] \arrow[dr, phantom, "\square"] & \supset V_K \arrow[d] \\
\mathrm{Spm}\, \mathcal{O}_K = S \arrow[r, "\text{induced by } \Delta_X^{\log}"'] & P & \supset U_K
\end{tikzcd}$$

$X * S.$

algebraic construction
corresponding to
shrinking the radius

$$D^n = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}\left(\mathcal{O}_K[X_1, \ldots X_n], \mathcal{O}_{\bar{K}}\right)$$

$$\cup$$

$$f^{-1}(0) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg}}\left(\mathcal{O}_K[X_1, \ldots X_n]/(f_1, \ldots f_n), \mathcal{O}_{\bar{K}}\right)$$

$$\begin{array}{ccc} f^{-1}(0) & & \mathcal{O}_K[X_1, \ldots X_n]/(f_1, \ldots f_n) \\ \parallel & & \parallel \\ G & & \mathcal{O}_L \end{array}$$

$r > 0$ rational number

K'/K finite separable ext

$e = e_{K'/K}$ ramification index

assume $\underline{e \cdot r}$ ~~is~~ integer

$S' = \mathrm{Spec}\, \mathcal{O}_{K'}$ $\quad P_{S'} = P \times_S S'$

$P^{(r)}_{S'}$ blow up of $P_{S'}$ at

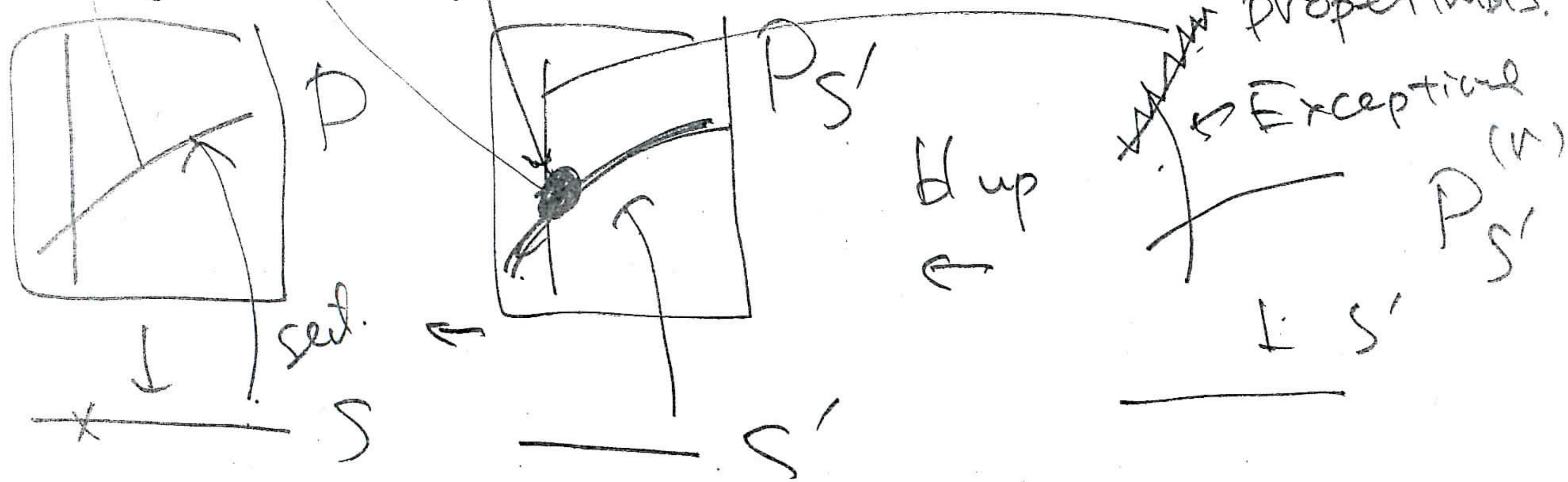
$$S_2 - \mathcal{O}_{K'}/m_{K'}^{ev} \longleftrightarrow \underline{S'} = S_1 - \mathcal{O}_{K'}$$

$$\underline{S'} \cap P_{S'} \quad \leftarrow \text{induced by } \underline{\Delta_X^{log}}$$

$$\boxed{(U_1 = 1,\ S_i = T_i,\ i = 2, \dots, d,\ (\pi'^{ev}))}$$

remove the proper transform of the closed fiber

$U_1 = 1$
$S_i = T_i$, $i = 2, \dots, d$



Example

$$X = \mathbb{A}^d_k = \operatorname{Spec} k[T_1, \ldots, T_d]$$

$$\cup$$

$$D = (T_1 = 0)$$

$$K = k(T_2, \ldots, T_d)((T_1))$$

$$X \times_k S = \operatorname{Spec} \mathcal{O}_K[S_1, \ldots, S_d] \leftarrow X * S = P$$

$$P = \operatorname{Spec} \mathcal{O}_K[U_1^{\pm 1}, S_2, \ldots, S_d]$$

$$S_1 = U_1 T_1 \qquad U_1 = \frac{S_1}{T_1}$$

$$P_{S'} = \mathrm{Spf}\, \underbrace{\mathcal{O}_{K'}[U_1^{\pm 1}, S_2, \ldots, S_d]}_{A}$$

$$P^{(r)}_{S'} = \mathrm{Spf}\, \mathcal{O}_{K'}[V_1, \ldots, V_d]$$

$$= \mathrm{Spf}\, A\left[\frac{U_1 - 1}{\pi'^{\,e\cdot r}}, \frac{S_2 - T_2}{\pi'^{\,e r}}, \ldots, \frac{S_d - T_d}{\pi'^{\,e r}}\right]$$

uniformizer of $\mathcal{O}_{K'}$

$$\begin{tikzcd}
Q^{(r)}_{S'} \arrow[d] & V_{K'} \arrow[l, phantom, "\supset"] \arrow[d] \\
P^{(r)}_{S'} \arrow[d] & U_{K'} \arrow[l, phantom, "\supset"] \arrow[d] \\
S' & \mathrm{Spec}\, K' \arrow[l]
\end{tikzcd}$$

normalization

smooth always

$\exists\ \underline{K'}/K$ (1) $e \cdot r$ integer

(2) $L \subset K'$

(Epp's theorem) (3) $Q^{(r)}_{S'}$ has geometrically reduced closed fiber.

$\bar{F}$ algebraic closure of the residue field F of K

$$\mathcal{O}_{K'} \longrightarrow \bar{F}$$

$$Q^{(r)}_{\bar{F}} = Q^{(r)}_{S'} \times_{\mathrm{Sp}\,\mathcal{O}_{K'}} \mathrm{Sp}\,\bar{F} \quad \text{geometric fiber}$$

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(3) <u>reduced</u>.

$$\begin{tikzcd}
G \arrow[d, phantom, "\rotatebox{90}{$=$}"] & & & \\
\mathrm{Spm}(L\otimes K') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_{K'} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Q^{(r)}_{S'} \arrow[d] \arrow[ddr, phantom, "\square"] & Q^{(r)}_{\bar F} \arrow[l, "\cong"'] \arrow[dd] \\
\mathrm{Spm}\,K' \arrow[r, "\varphi"'] & U_{K'} \arrow[r, hook] \arrow[d, "\text{induced by } \Delta_X^{\log}"'] & P^{(r)}_{S'} \arrow[d] & \\
& \mathrm{Spm}\,K' \arrow[r, hook] & S' & \mathrm{Spec}\,\bar F \arrow[l]
\end{tikzcd}$$

reduction map

$$G \longrightarrow Q^{(r)}_{\bar F}$$

$\underline{\underline{G^r_{\log}}} = \{\sigma \in G \mid$ image of σ in $Q^{(r)}_{\bar F}$ lies in the same $\underline{\text{conn. cpt}}$ as the image of id $\}$

Example $\quad p = \operatorname{char} K = \operatorname{char} k$

K as before Artin-Schreier ex

$$\operatorname{Hom}(G_K, \mathbb{Z}/p\mathbb{Z}) = H^1(K, \mathbb{Z}/p\mathbb{Z})$$

$$\cup \quad \chi_a \longleftarrow a \in K / (a^p - a,\ a \in K)$$

$$\mathrm{Fil}^n = \text{Image of } m_K^{-n}$$

$a \in m_K^{-n}$ s.t. $\chi_a \in \mathrm{Fil}^n, \notin \mathrm{Fil}^{n-1}$

$L/K \quad t^p - t = a.$

$$G = \mathrm{Gal}(L/K) = \mathbb{Z}/p\mathbb{Z}$$

$$G^r_{\log} = 0 \iff r > \underline{\underline{n}}$$

~~Grade~~ Graded pieces (on last piece of fil.)

$$G = \mathrm{Gal}(L/K) \supset G^r_{\log}$$

$$G^s_{\log} = 1 \iff s > r \text{ : last jump}$$

$G^r_{\log}$ is abelian & killed by p.

$\bar{F}$, $\mathcal{O}_K = \hat{\mathcal{O}}_{X,\xi} \supset \mathcal{O}_{X,\xi}$

rsw: $\mathrm{Hom}(G^r_{\log}, \mathbb{F}_p) \hookrightarrow$ $\bar{F}$-vect. sp. of dim 1

↑ canonical injection

$\mathrm{Hom}_{\bar{F}}(m^r_{\bar{K}}/m^{r+}_{\bar{K}}, \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} \bar{F})$

locally fr on X

↑ $\bar{F}$-vector space

$\bar{K}$ = sep closure of K.

$m^r_{\bar{K}} = \{a \in \bar{K} \mid v_K(a) \geq r\}$ $v_K(\pi_K) = 1$ unif of K

$m^{r+}_{\bar{K}} = \{ \text{——} \mid v_K(a) > r\}$