

AWS 2012

Introduction to wild ramification
of schemes & sheaves

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- ✓ 1. Formula for the Euler number
2. Filtration by ramification groups
3. Blow-up & characteristic class

1 Euler number

k field. $p = \text{char } k$.

U (Smooth) separated scheme of finite type
 $/k$.

l prime $\neq p$.

\Rightarrow smooth l -adic sheaf on
 the étale site of X

smooth ℓ -adic sheaf \mathcal{F}



(U connected)

geom. pt

ℓ -adic rep'n of $\pi_1(U, \bar{x})$

algebraic fundamental gp.

pro finite

a gp of $\mathbb{A}_\ell = \text{Gal}(\bar{K}/K)$

K : function field of U .

$H_c^i(U_{\mathbb{R}}, \mathbb{Z})$ compact support
cohomology

finite dimensional \mathbb{Q} -vect. space

= 0 except for $0 \leq i \leq 2 \cdot \dim U$.

$$\chi_c(U_{\mathbb{R}}, \mathbb{Z}) = \sum_{i=0}^{2 \cdot \dim U} (-1)^i \dim H_c^i(U_{\mathbb{R}}, \mathbb{Z})$$

Euler #

2. Grothendieck - Ogg - Shafarevich
formula.

($\dim U = 1$) k perfect
smooth curve

$$\chi_c(U_{\bar{k}}, \mathbb{Z}) = \text{rk } \mathbb{Z} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell)$$

($\dim k = 0 \implies = 0$)

$$D > 0$$

G-O-S formula

$$\chi_c(U_{\mathbb{F}_2}, \mathcal{F}) - \text{rk } \mathcal{F} \cdot \chi_c(U_{\mathbb{F}_2}, \mathcal{O}_U)$$

smooth sheaf
on U

$$= - \sum_{x \in X \setminus U} \frac{\text{Sw}_x \mathcal{F}}{x} \cdot \text{deg } x$$

X smooth compactification

Swan
conductor

proper

of $U \hookrightarrow X$
smooth/ \mathbb{F}_2 dense open immersion

$\in \mathbb{N}, \geq 0$

Conductor

$$U \hookrightarrow X \ni x$$

$$\hat{\mathcal{O}}_{X,x} \quad \begin{array}{l} \text{complete} \\ \text{d.v.v} \end{array}$$

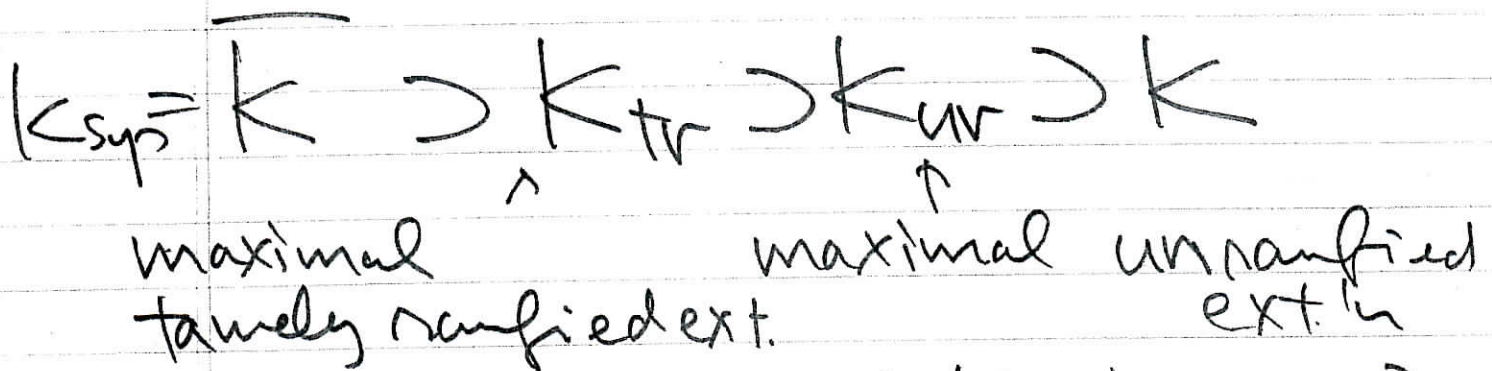
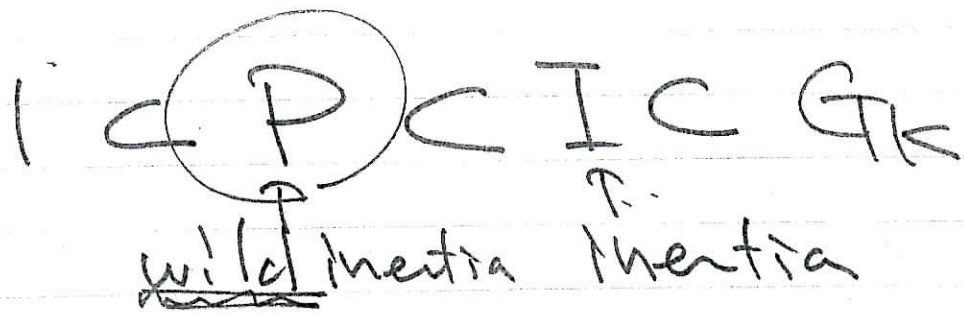
$$K = K_x = \text{Frac } \hat{\mathcal{O}}_{X,x} \quad \begin{array}{l} \text{local field} \\ \text{at } x \end{array}$$

$$\frac{\pi_1(U, \bar{x})}{\cong} \leftarrow \begin{array}{l} G_{K_x} = G_{\bar{K}} \\ \cong \\ G_{\bar{K}_x} \end{array}$$

ℓ -adic rep'n V

$$\downarrow \\ \mathbb{Z}/\ell^n$$

$$\frac{\text{Gal}(\bar{K}_x/K_x)}{\cong}$$



$$K_{\text{tr}} = K_{\text{ur}}(\pi_{K_{\text{ur}}} | P_{K_{\text{ur}}})$$

\uparrow
 uniformizer of K

$$I/P \cong \frac{I/P}{P_{K_{\text{ur}}}} \cong \pi \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$I \cong P \rtimes I/P$$

\uparrow
 pro- p Sylow gp of I

ramification gps

① upper numbering

- lower numbering

$$P \supseteq G_{TK, \log}^r \quad r > 0, r \in \mathbb{Q}$$

↑
closed normal subgroups

decreasing filtration

$$P = \bigcup_{r > 0} G_{TK, \log}^r \quad \curvearrowright \quad V$$

↑
prop

$P \neq Q \Rightarrow$ acts through finite gT

ℓ -adic rep

∃! decomposition

$$V = \bigoplus_{\substack{r \geq 0 \\ r! \in \mathbb{Q}}} V^{(r)}$$

slope
dec'n

- ~~$V^{(r)}$~~ stable by G_K
 - G_K^s acts trivially on $V^{(r)}$
- \Leftrightarrow $s > r$

$$\text{Sw}_K V = \sum_{r \in \mathbb{Q}} r \cdot \dim V^{(r)}$$



$(\Rightarrow) = 0 \Leftrightarrow P \text{ acts trivially on } V.$

Measure of wild ramification

lower numbering

for simplicity assume

G_K acts on V via finite $g \leq G$

$\rightarrow G = \text{Gal}(L/K)$ L finite Galois K

$$G_i = \ker \left(G \longrightarrow L^{\times} / (1 + \mathfrak{m}_L^i) \right)$$

$$i \in \mathbb{N}, \geq 1.$$

~~★~~ Swan character

$$S_{L/K}(\sigma) \quad \sigma \in G, \supset I \text{ image of } I \subset G_K$$

$$= 0 \quad (\text{if } \sigma \notin I)$$

$$\stackrel{\text{---}}{=} \sum_{\pi_L} \left(\frac{\sigma(\pi_L)}{\pi_L} - 1 \right) \quad \pi_L \text{ unif. of } L$$

$$\sigma \in I, \neq 1$$

normalized
 discrete valuation

$S_{L/K}(\sigma)$ is def'd by requiring

1-14

$$\sum_{\sigma \in G} S_{L/K}(\sigma) = 0$$

Swan character is a character
of a rep'n of G . (Fact)

$$S_{L/K} V = \frac{1}{|I|} \sum_{\sigma \in I} S_{L/K}(\sigma) \cdot \text{Tr}(\sigma: V)$$

$\in \mathbb{N}$.

Generalization to higher dimension

U smooth ~~set~~ sep sch of $f.t/l_k$

$d = \dim U$. arbitrary

\Rightarrow smooth l -adic sheaf

$$\chi_c(U_{\bar{k}}; \mathcal{F}) - v_k \mathcal{F} \cdot \chi_c(U_{\bar{k}, \mathbb{Q}_l}; \mathcal{O}_l) = ?$$

Swan class

$$Sw_U \cong \frac{CH_0(X \setminus U)}{\mathbb{Q}(S_{p^0})}$$

p-powerth
rt of 1

generalization of the Swan conductor

X compactification of U.

proper / ts $\supset U$
 dense
 open

CH₀(S) Chow gp of 0-cycles

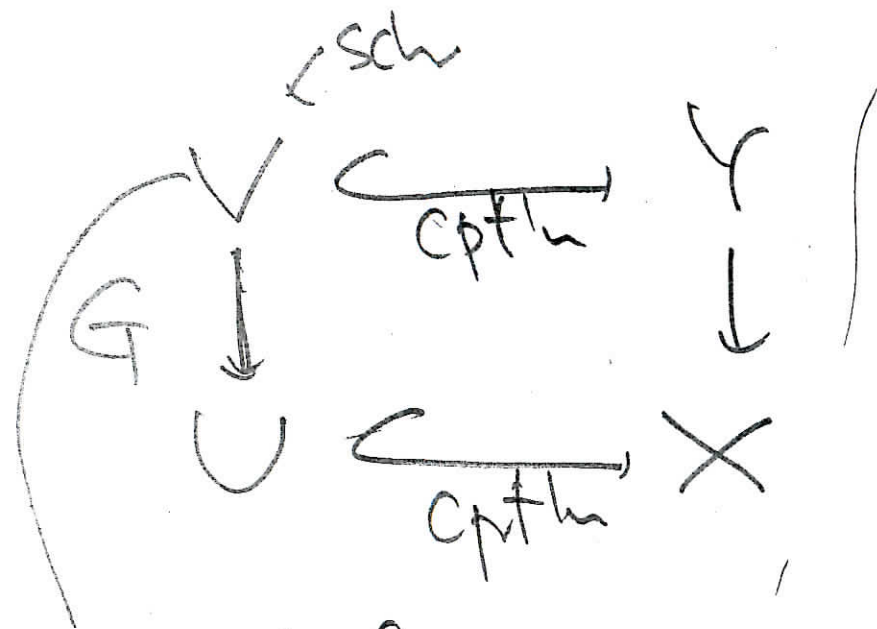
$$= \left(\bigoplus_{S \in \mathcal{S}} \mathbb{Z} \cdot [s] \right) / \text{rational equivalence}$$

S: closed pts

$$\underline{S_{WU}(\mathcal{X})} = \sum_{\sigma \in G} \underline{S_G(\sigma)} \cdot \text{Tr}(\sigma: V)$$

Simplifying assumpt'n
 $\pi_1 \rightarrow G \curvearrowright V$
 finite gr. = rep.

ℓ-adic rep'n
 of $\pi_1(U, \bar{x})$
 corrsp. to \mathcal{X}



finite étale
Galois covering

Simplifying assumption.

Y smooth $\dim d$

$$V = Y - D$$

divisor of Y with
Simple normal crossing

$$D = \cup_i D_i \quad \text{smooth}$$

$D_i \cap \dots \cap D_j$ transversal.

$S_G(\sigma) =$ logarithmic modification of $\frac{Y \times Y}{\Delta_Y}$

Intersection product (σ, Δ_Y)
available in normal