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Introduction to wild ramification of schemes & sheaves

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✓ 1. Formula for the Euler number

2. Filtration by ramification groups

3. Blow-up & characteristic class

1 Euler number

k field. $p = \operatorname{char} k$.

U (smooth) separated scheme of fin type $/k$.

ℓ prime $\neq p$.

$\mathcal{F}$ smooth ℓ-adic sheaf on the étale site of X

smooth ℓ-adic sheaf $\mathcal{F}$

$\updownarrow$ (U connected)

ℓ-adic rep'n of $\underline{\pi_1(U, \bar{x})}$ geom. pt

algebraic fundamental gp.

pro finite

a qt of $G_K = \mathrm{Gal}(\bar{K}/K)$

K : function field of U.

$H^i_c(U_{\bar{k}}, \mathcal{F})$ compact support cohomology

finite dimensional $\mathbb{Q}_\ell$-vect. space

$= 0$ except for $0 \le i \le 2 \cdot \dim U$.

$$\chi_c(U_{\bar{k}}, \mathcal{F}) = \sum_{i=0}^{2 \cdot \dim U} (-1)^i \dim H^i_c(U_{\bar{k}}, \mathcal{F})$$

Euler #

2. Grothendieck - Ogg - Shafarevich formula.

(dim $\underset{\sim}{U} = 1$) k perfect

smooth curve

$$\chi_c(U_{\bar{k}}, \mathcal{F}) - \mathrm{rk}\,\mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell)$$

($\dim k = 0 \Rightarrow \quad = 0$)

$p > 0$

G-O-S formula

$$\chi_c(U_{\bar{k}}, \mathcal{F}) - \mathrm{rk}\,\mathcal{F} \cdot \chi_c(U_{\bar{k}}, \mathbb{Q}_\ell)$$

smooth sheaf on U

$$= - \sum_{x \in X \setminus U} \underline{\mathrm{Sw}_x \mathcal{F}} \cdot \deg x$$

X smooth compactification of $U \hookrightarrow X$

proper smooth $/k$ — dense open immersion

Swan conductor $\in \mathbb{N}, \geq 0$

Conduction

$$U \hookrightarrow X \ni x$$

$\hat{\mathcal{O}}_{X.x}$ complete d.v.r

$K = K_x = \operatorname{Frac} \hat{\mathcal{O}}_{X.x}$ local field at x

$$\pi_1(U, \bar{x}) \leftarrow G_{K_x} = G_K$$

$$G_{K_x} = \operatorname{Gal}(\bar{K}_x / K_x)$$

$\pi_1(U, \bar{x})$ ↻ V l-adic rep'n

$\updownarrow$

$\mathcal{F}/U$

$$1 \subset P \subset I \subset G_K$$

wild inertia   inertia

$$K_{sep} = \overline{K} \supset K_{tr} \supset K_{ur} \supset K$$

maximal tamely ramified ext.   maximal unramified ext'n

$$K_{tr} = K_{ur}(\pi^{1/m} \mid p \nmid m)$$

uniformizer of K

$$I/P \xrightarrow{\sim} \varprojlim_{p \nmid m} \mu_m \simeq \prod_{q \neq p} \mathbb{Z}_q$$

$$I \simeq P \rtimes I/P$$

pro-p Sylow gp of I

ramification gps

- upper numbering
- lower numbering

$$P \supset G^{r}_{K,\log} \quad r > 0,\ r \in \mathbb{Q}$$

closed normal subgroups

decreasing filtration

$$P = \bigcup_{r>0} G^{r}_{K,\log} \curvearrowright V$$

pro-p

l-adic rep

$p \neq l \Rightarrow$ acts through finite qt

$\exists 1$ decomposition

$$V = \bigoplus_{\substack{r \geq 0 \\ r \in \mathbb{Q}}} V^{(r)}$$ slope dec'n

- $V^{(r)}$ stable by G_K
- $G_{K,\log}^{s}$ acts trivially on $V^{(r)}$ $\Leftrightarrow$ $s > r$.

$$\mathrm{Sw}_K V = \sum_{r \in \mathbb{Q}} r \cdot \dim V^{(r)}$$

$$(\Rightarrow \quad = 0 \iff P \text{ acts trivially on } V.)$$

measure of wild ramification

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lower numbering

for simplicity assume

G_K acts on V via finite gt G

$G = \mathrm{Gal}(L/K)$ $\quad L$ finite Galois $/K$

$$G_i = \ker\left(G \longrightarrow L^\times / 1 + m_L^i\right)$$

$$i \in \mathbb{N},\ \geq 1.$$

Swan character

$S_{L/K}(\sigma)$ $\quad \sigma \in G \supset I$ image of $I \subset G_K$

$= 0$ (if $\sigma \notin I$)

$= v_L\left(\frac{\sigma(\pi_L)}{\pi_L} - 1\right)$ $\quad \pi_L$ unif. of L

$\sigma \in I, \neq 1$

normalized discrete valuation

$S_{L/k}(1)$ is def'd by requiring

$$\sum_{\sigma \in G} s_{L/k}(\sigma) = 0$$

Swan character is a character of a rep'n of G. (Fact)

$$\mathrm{Sw}_k V = \frac{1}{|I|} \sum_{\sigma \in I} s_{L/k}(\sigma) \cdot \mathrm{Tr}(\sigma : V)$$

$\in \mathbb{N}$.

Generalization to higher dimension

U smooth ~~sel~~ sep sch of f.t / k

$d = \dim U$ arbitrary

$\mathcal{F}$ smooth ℓ-adic sheaf

$$\chi_c(U_{\bar k}, \mathcal{F}) - \mathrm{rk}\,\mathcal{F} \cdot \chi_c(U_{\bar k}, \mathbb{Q}_\ell) = ?$$

Swan class

$$\mathrm{Sw}_U \mathcal{F} \in \underline{\underline{CH_0(X \setminus U)}}_{\mathbb{Q}(\zeta_{p^\infty})}$$

p-power-th rt of 1

generalization of the Swan conductor

X compactification of U.

proper / k $\supset$ U

~~dense~~ open

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(17)
2- -

$\underline{CH_0(S)}$ Chow gp of 0-cycles

$$= \left(\bigoplus_{s \in S} \mathbb{Z}\cdot[s] \Big/ \text{rational equivalence} \right) \Big/ \mathbb{Q}(\zeta_{p^\infty})$$

s: closed pts.

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$$\underline{Sw_U(\mathcal{F})} = \sum_{\sigma \in G} s_G(\sigma) \cdot \mathrm{Tr}(\sigma : V)$$

Simplifying assump'n

$\pi_1 \to G \curvearrowright V$

finite gt. $\cong$ rep'n

ℓ-adic rep'n of $\pi_1(U, \bar{x})$ corresp. to $\mathcal{F}$

1–18
2–

$$
\begin{tikzcd}
V \arrow[r, hook, "\text{cpt'n}"'] \arrow[d, "G"'] & Y \arrow[d] \\
U \arrow[r, hook, "\text{cpt'n}"'] & X
\end{tikzcd}
$$

< Sch

finite étale Galois covering

Simplifying assump'n.

Y smooth dim d

$V = Y \setminus D$

divisor of Y with simple normal crossing

$D = \bigcup_i D_i$ smooth

$D_{i_1} \cap \cdots \cap D_{i_j}$ transversal.

$-S_G(\sigma) =$ logarithmic modification of $Y \times Y$

intersection product $(\Gamma_\sigma, \Delta_Y)$

graph of σ — diagonal