<u>Aim:</u> $F = k((t))(x)$ $\quad$ k alg. closed

G finite group s.th. $\mathrm{char}(k) \nmid |G|$

and s.th. all Sylow subgroups of G

are abelian metacyclic

Want to construct a G-crossed

product division algebra over F.

Q: Can we patch division algebras?

No ...

$$\left.\begin{array}{l} G\text{-Galois field ext}/F \rightsquigarrow G\text{-Galois } F\text{-algebras} \\ \\ F\text{-division algebras} \rightsquigarrow F\text{-csa's}. \end{array}\right\} (*)$$

(*) have patching theorems

Strategy / Setup:

$$G = \langle P_i \mid i = 1, \ldots, r \rangle$$

P_i p_i - Sylow subgroups of G

Pick pts Q_i $(i = 1, \ldots, r) \in \mathbb{P}^1_k = X$

w.l.o.g. finite

$F_i := F_{\{Q_i\}}$ $i = 1, \ldots, r$

$F_{r+1} := F_U$ where $U = X \setminus \{Q_1 \ldots Q_r\}$

$F_0 = F_\emptyset$

Issues:

- find "building blocks" over each $\mathcal{F}_i$ $i=1,\dots,r+1$

- make sure they agree on $\mathcal{F}_\emptyset$

  by making them trivial there

- make sure the result of patching is division

Building blocks:

Fix $i \in \{1, \dots, r\}$, $Q := Q_i$, $P := P_i$, $p := p_i$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ p-Sylow subgroup

P abelian of rk ≤ 2 : $P = C_q \times C_s$

Q defined by $t = 0$, $x = c$ some $c \in k$.

First: Construct P-Galois ext. of $F_{\{Q\}}$.

Pick $a, b \in F_{\{Q\}}$ s.th. $L = F_{\{Q\}}(y, z)$

where $y^q = a$, $z^s = b$ has Galois

group $P = C_q \times C_s$

e.g. $a = \dfrac{x-c}{x-c-t}$, $b = \dfrac{x-c-t^2}{x-c-t-t^2}$

$$\begin{tikzcd}[arrows=dash]
 & L \arrow[dl] \arrow[dr] & \\
F_{\{Q\}}(y) \arrow[dr] & & F_{\{Q\}}(z) \arrow[dl] \\
 & F_{\{Q\}} &
\end{tikzcd}$$

$\rightsquigarrow$ correct Galois group

$\zeta \in k$ primitive $|P|$-th root of unity

(ex. since $\mathrm{char}(k) \nmid |P|$)

Define D by : generators Y, Z

relations: $Y^s = y$, $Z^q = z$

($\rightsquigarrow$ $Y^{|P|} = a$, $Z^{|P|} = b$)

$YZ = \zeta Z Y$

Check: this is a csa. (symbol algebra)

Moreover: $yz = Y^s Z^q = \zeta^s Z Y^s Z^{q-1} = \underbrace{(\zeta^s)^q}_{1} \underset{z}{Z^q} \underset{y}{Y^s}$

Hence $L \hookrightarrow D$ as a subfield (maximal)

Need to check: D is division

To see this, show that D is a cyclic algebra.

<u>Known:</u> If v is a valuation on a field E which extends uniquely to a cyclic ext. $\hat{E}/E$, deg n and if $v(a)$ has order n in $v(E)/nv(E)$ then for any $\sigma \in \mathrm{Gal}(\hat{E}/E)$, $(a, \hat{E}/E, \sigma)$ is division.

Want : D, L become "trivial" over F_ϕ

for L : $L \otimes_{F\{Q\}} F_\phi \cong F_\phi^{|P|}$ "split ext."

for D : $D \otimes_{F\{Q\}} F_\phi \simeq Mat_{|P|}(F_\phi)$ "split alg."

OK (check directly since $F_\phi = k(x)((t))$)

Induced algebras:

G finite group, $H \leq G$

$L|F$ is H-Galois field extension

Define $\mathrm{Ind}_H^G L$ as follows:

- $c_1, \dots, c_m$ coset reps of H in G
- $\mathrm{Ind}_H^G L := \bigoplus_{i=1}^{m} L$ as vector space

  with "standard basis" $c_1, \dots, c_m$

• fix i : $g c_i = c_j h$ some $h \in H$

$$\Rightarrow g = c_j h c_i^{-1}$$

define $g(c_i \cdot a) = c_j h(a)$

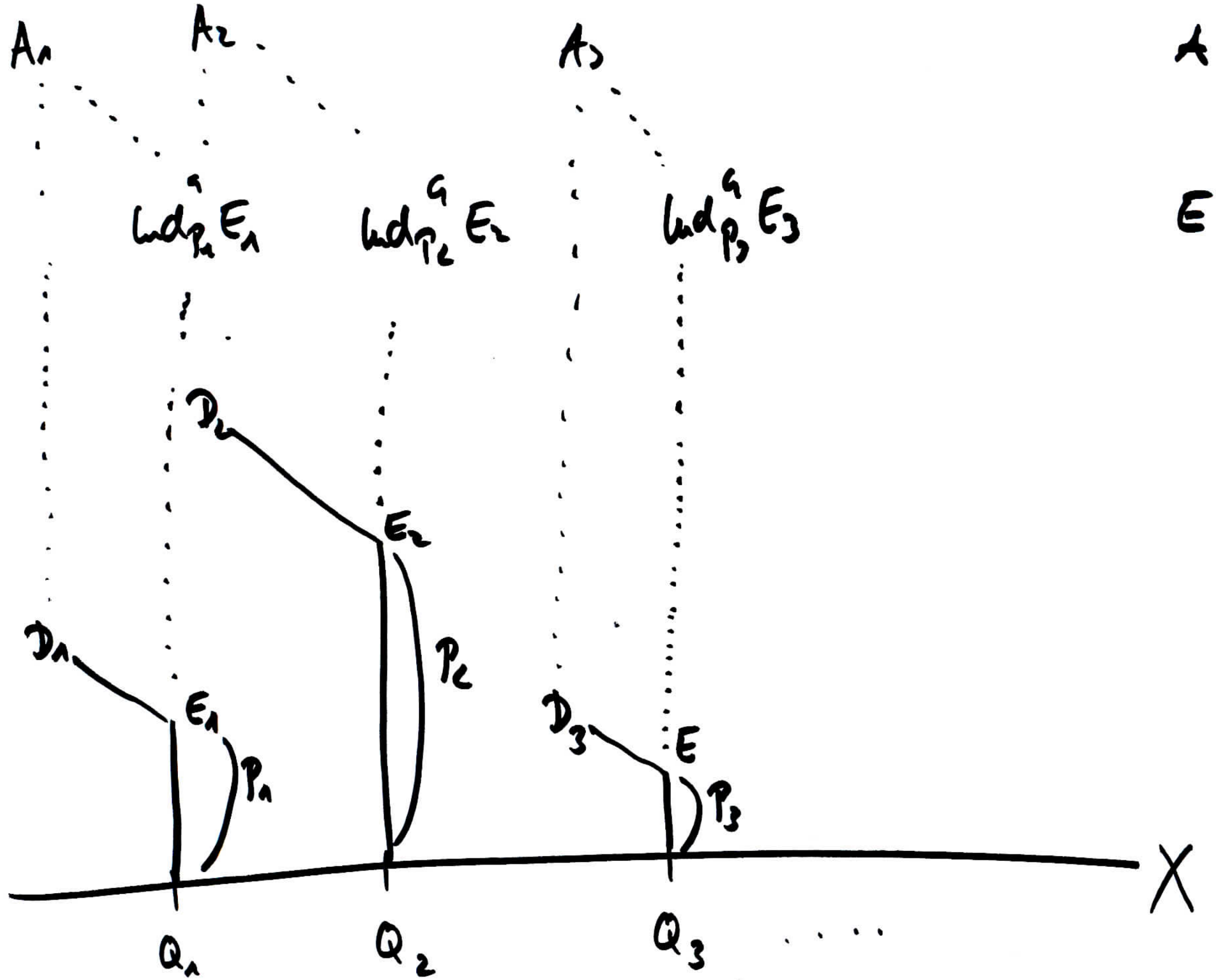
This defines F-linear G-action.

Easy to show: $\mathrm{Ind}_H^G L$ is a G-Galois F-alg.

$D \mid F$ H-crossed product alg.

$\mathrm{Mat}_{|G/H|}(D) = A$

Check: Splitness properties are preserved.


A1
A2
A3
A
Ind^G_{P1} E1
Ind^G_{P2} E2
Ind^G_{P3} E3
E
D2
E2
P2
D1
E1
P1
D3
E
P3
X
Q1
Q2
Q3
. . . .

Patching gives:

.) a G-Galois F-algebra $E|F$

.) a CSA A/F

.) One can show:

$$
\begin{tikzcd}
A_i \otimes_{F_i} F_\emptyset \arrow[r] & A_\emptyset = \mathrm{Mat}_{|G|}(F_\emptyset) \\
\mathrm{Ind}_{P_i}^{G} E_i \otimes_{F_i} F_\emptyset \arrow[u] \arrow[r] \arrow[ur, phantom, "//"] & F_\emptyset^{|G|} \; (= E_\emptyset) \arrow[u]
\end{tikzcd}
$$

$\Rightarrow$ get inclusion $E \hookrightarrow A$, maximal for degree reasons

RTS: (E is a field $\Leftrightarrow$ and) A is division

Let D be a division algebra in the class of A:

$$|P_i| = [E_i : F_i] = \deg(D_i) \mid \deg D \quad (= \operatorname{ind} A)$$

$$\deg(A) = |G| = \operatorname{lcm}(|P_i|,\ i=1,\dots,r) \mid \deg D \mid \deg A$$

$$\Rightarrow \deg D = \deg A \quad \Rightarrow A \text{ division}$$

$$\Rightarrow E \text{ field}$$

$\square$

Q: Can we patch division algebras?

Maybe ....

$$\begin{tikzcd} & F_0 & \\ F_1 \arrow[ur, dash] & & F_2 \arrow[ul, dash] \\ & F \arrow[ul] \arrow[ur, dash] & \end{tikzcd}$$

Show: $$\mathrm{Br}(F) \longrightarrow \mathrm{Br}(F_1) \times_{\mathrm{Br}(F_0)} \mathrm{Br}(F_2)$$

is a group isomorphism

(using patching)

Patching leads to <u>local-global principles</u>.