Patching $\Longleftrightarrow$ 1, 2

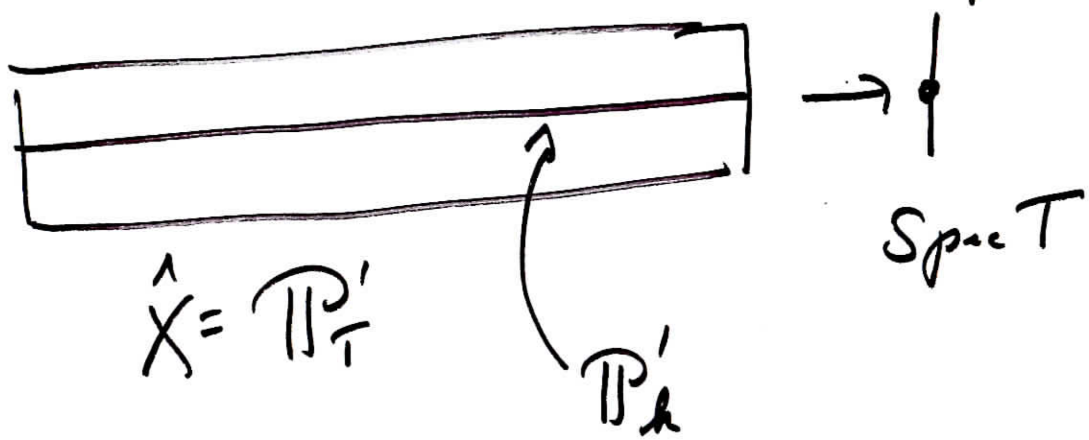
K cdvf e.g. $\mathbb{Q}_p$, $k((t))$

$\cup$

T cdvr $\mathbb{Z}_p$, $k[[t]]$

F 1-vbl fn fld / K e.g. $K(x)$

$\hat{\mathbb{P}}^1_T$ $\mathbb{P}^1_K$



$F = k((t))(x)$

$U \subset X \qquad \hat{X}$, F = rat'l fns on $\hat{X}$

R_U = {rat'l fns in F regular on U} $\subset F$

reg at gen pt of X

$\hat{R}_U$ = t-adic completion of R_U

F_U = frac $\hat{R}_U$

Ex. $U_1 \subset X = \mathbb{P}^1_k$ (delete ∞)

$\quad \| $

$\mathbb{A}^1_k$, affine x-line

$\quad \|$

Spec $k[x]$

$\hat{R}_{U_1} = k[x][[t]]$

F_{U_1} = frac $\hat{R}_{U_1}$

U_2 : delete $x=0$

$\| $

x^{-1}-line, another $\mathbb{A}^1_k$

$\|$

Spec $k[x^{-1}]$

$\hat{R}_{U_2} = k[x^{-1}][[t]]$

$F_U = \mathrm{frac}(\cdot \; - \cdot \,)$

U_0 : delete $x = 0, \infty$ $\qquad U_1 \cap U_2$

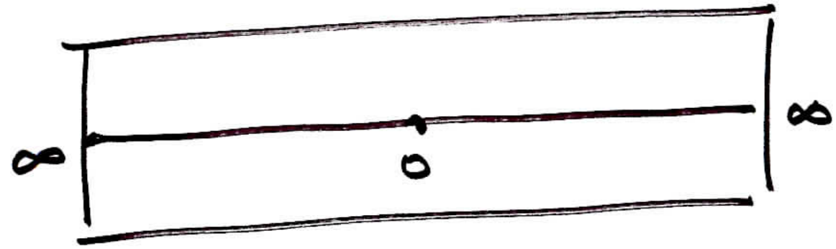
$\|$

Spec $k[x, x^{-1}]$

$\hat{R}_{U_0} = k[x, x^{-1}][[t]]$

$F_{U_0} = \mathrm{frac}(\text{———})$

$\hat{X}$

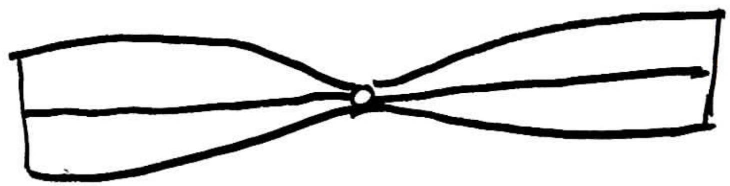


F

Spec $\hat{R}_{U_1}$

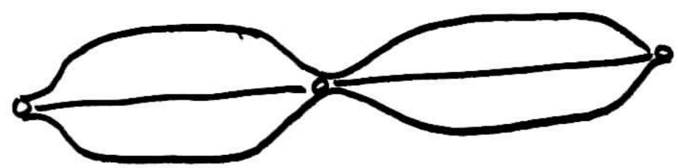
F_1

Spec $\hat{R}_{U_2}$



F_2

Spec $\hat{R}_{U_3}$



F_0

$$\begin{tikzcd} & F_0 \arrow[dl, dash] \arrow[dr, dash] & \\ F_1 \arrow[dr, dash] & & F_2 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

Have patching!

viz. (1), (2) hold

why?

(1) $A_0 \in GL_n(F_0)$ want $A_i \in GL_n(F_i)$ $i=1,2$

$$A_0 = A_1^{-1} A_2$$

Key ingredient: <u>additive</u> decomp.

Ex. U_1 $k[x]$

U_2 $k[x^{-1}]$

U_0 $k[x, x^{-1}]$ add. decomp

Ex: $k[x, \frac{1}{x}]$, $k[x, \frac{1}{x-1}]$, $k[x, \frac{1}{x}, \frac{1}{x-1}]$

partial fraction decomp.

For ① :

Special case:

$A_0 \in GL_n(\hat{R}_0)$, $A_0 \equiv I \pmod t$

$1 + (a+b)t = (1+at)(1+bt) + \text{h.o.t.}$

Ex. $n = 1$ $\quad A_0 = (1 + (x^2 + 2 + x^{-1})t)$

$\in GL_1(\hat{R}_0) \quad \hat{R}_0^{\times}$

Mod t $\quad A_0 = 1 = 1 \cdot 1 \pmod t$

Mod t^2 $A_0 = (1 + (x^2+2)t)(1 + x^{-1}t) \mod t^2$

$\boxed{(x + 2x^{-1})t^2}$

Mod t^3

$A_0 = (1 + (x^2+2)t - xt^2)(1 + x^{-1}t - 2x^{-1}t^2) \mod t^3$

$\vdots$

$n \times n$ case $\qquad A_0 \equiv I \pmod{t}$

similar

Get (1), in special case.

For general case, use:

Weierstrass Preparation Thm

Thm If $U \subseteq X$ then every $f \in F_U$ can be written as

$f = au, \qquad a \in F, \quad u \in \hat{R}_U^{\times}.$

W. Prep: — re proof.

$T = k[[t]]$, $U = \mathbb{A}^1_k$, x-line

$F_U = \text{frac } \hat{R}_U$. WMA $f \in \hat{R}_U$.

$$\hat{R}_U = k[x][[t]]$$

$$f = \sum_{i=0}^{\infty} f_i t^i \qquad f/f_0 \in k(x)[[t]] = \hat{R}_\emptyset = \hat{R}_{U_0}$$

$f/f_0 \equiv 1 \pmod t$

$U_1 = \{\infty\}$ $\qquad U_2 = U$

$U_1 \cup U_2 = X$, $U_1 \cap U_2 = U_0 = \emptyset$

$|x|$ case of Cartan (spec. case)

$f/f_0 = f_1^{-1} f_2 \qquad f_1 \in \hat{R}_{v_1}^{\times}, \quad f_2 \in \hat{R}_{v_2}^{\times}$

$v_2 \wr v$

$a := f_0 f_1^{-1} = f f_2^{-1}$

$f_0 f_1^{-1} \in \hat{R}_{w_1}[x] = k[x^{-1}]((t))[x]$

$f f_2^{-1} \in \hat{R}_v = k[x]((t))$

no x^{-1}

bdd in x

$a \in k((t))(x) \subseteq F$

$u = f_2 \in \hat{R}_v^{\times}$

$f = au \qquad a \in F, \quad u \in \hat{R}_v^{\times}$

Want (2) $F = F_1 \cap F_2 \subseteq F_0$

$$\left(\begin{array}{ll} U_1, U_2 \subseteq X & U_0 = U_1 \cap U_2 \\ & U = U_1 \cup U_2 \\ \Rightarrow F_U = F_{U_1} \cap F_{U_2} \subseteq F_{U_0} & \end{array} \right)$$

Use Weierstrass.

$T = k((t))$, $U_1 =$ aff. x^{-1}-line

$U_2 =$ aff. x-line

$\hat{R}_{U_1} = k(x^{-1})((t))$ $\quad \hat{R}_{U_2} = k(x)((t))$

$R' := \hat{R}_1(x) \cap \hat{R}_2 = k((t))(x) \leftarrow$ frac $= F$

$f \in F_1 \cap F_2$ WTS $f \in F = \mathrm{frac}\, R'$

Weier Prep

$$f = f_1 u_1 = f_2 u_2$$

$f_1 \in F,\ u_1 \in \hat{R}^\times_{U_1}$

$f_2 \in F,\ u_2 \in \hat{R}^\times_{U_2}$

$a_i, b_i \in R'$

$i = 1,2:\ f_i = a_i / b_i$

$$f = a_1 u_1 / b_1 = a_2 u_2 / b_2$$

$$\underset{\hat{R}_1(x)}{a_1 b_2 u_1} = \underset{\hat{R}_2}{a_2 b_1 u_2} \in R' \in F$$

$$f = \frac{a_1 b_2 u_1}{b_1 b_2} \in F.$$

(1) in gen'l :

Use (2) to reduce to $U_1 \cap U_2 = \emptyset$

$\hat{R}_\emptyset = k(x)[[t]]$ cdvr

Reduce to prev case, $\equiv I \pmod{t}$.

Get (1) ✓.

$\therefore$ Get patching.

Can generalize:

- not just for line
  - smooth curve / T
  - even singular
- OK for other cdvf's.
- OK for more fields than

$$\begin{tikzcd}[column sep=small] & F_0 \arrow[dl, dash] \arrow[dr, dash] & \\ F_1 \arrow[dr, dash] & & F_2 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

$U_0 \quad U_1, U_2$

$U_1, \dots, U_n$ union $= X$

$U_i \cap U_j = U_0$ ind. of $i \neq j$

Application to Galois thy:

K cdvf, G fin gp.

$\Rightarrow$ G is a Galois gp $/ K(x)$.

<u>Idea</u> $\hat{X} = \mathbb{P}^1_T$ $\quad$ T cdvr

$X = U_1 \cup U_2$ $\quad$ $U_0 = U_1 \cap U_2$

$G = \langle H_1, H_2 \rangle$ $\quad E_i$

H_i is Gal. gp. $/ F_{U_i}$

$E_i \otimes_{F_{U_i}} F_{U_0}$ trivial$/ F_{U_0}$

$\qquad\qquad \parallel$

$\qquad\qquad F_{U_0}^{\oplus n_i}$

$\oplus E_i$: G-Gal. alg $/ F_{U_i}$ $\quad$ fld ext.

Patch: G-Gal alg $/F$ $\quad \equiv E/F$.