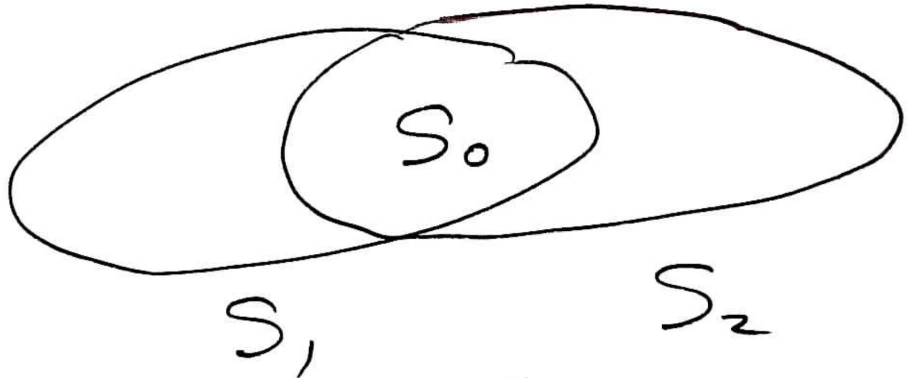
Patching



S

given structures

$$\begin{tikzcd}
 & F_0 \arrow[dl, dash] \arrow[dr, dash] & \\
F_1 \arrow[dr, dash] & & F_2 \arrow[dl, dash] \\
 & F &
\end{tikzcd}$$

F

want str.

Alg. approaches

Zariski, Grothendieck
formal scheme Raynaud

Tate, rigid analytic space

$\rightsquigarrow$ Galois thy
branched cover

DH '80's Inv. Gal. problem for $\mathbb{Q}_p(x) \leftarrow$
$k((t))(x)$
cdvf

J. H. & D. H. — Patching over fields

$$\begin{tikzcd}[column sep=small] & F_0 \arrow[dl, dash] \arrow[dr, dash] & \\ F_1 \arrow[dr, dash] & & F_2 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

1st: FDVS

"structure"

Vect(E) FDVS $/E$

$E \subseteq E'$

$$\mathrm{Vect}(E) \longrightarrow \mathrm{Vect}(E')$$

$$V \longmapsto V_{E'} = V \otimes_E E'$$

$$\begin{tikzcd}[column sep=small] & F_0 & \\ F_1 \arrow[ur, dash] & & F_2 \arrow[ul, dash] \\ & F \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

V_1 V_2

$$\mathrm{Vect}(F) \longrightarrow \mathrm{Vect}(F_i) \qquad i = 0, 1, 2$$

$$\mathrm{Vect}(F_i) \longrightarrow \mathrm{Vect}(F_0) \qquad i = 1, 2$$

<u>Patching problem</u>:

$$V_1 \otimes_{F_1} F_0 \xrightarrow[\mu]{\sim} V_2 \otimes_{F_2} F_0$$

<u>Sol'n</u>: FDVS V / F

together with F_i-vs isos

$$V \otimes_F F_i \xrightarrow[\iota_i]{\sim} V_i \qquad i = 1, 2$$

that are compat with

$$\begin{tikzcd}
(V \otimes_F F_1) \otimes_{F_1} F_0 \arrow[r, "\sim"] \arrow[d, "\iota_1 \otimes 1"', "\wr"] & V \otimes_F F_0 \arrow[r, "\sim"] & (V \otimes_F F_2) \otimes_{F_2} F_0 \arrow[d, "\wr"', "\iota_2 \otimes 1"] \\
V_1 \otimes_{F_1} F_0 \arrow[rr, "\sim", "\mu"'] & & V_2 \otimes_{F_2} F_0
\end{tikzcd}$$

$$\mathrm{Vect}\,F \xrightarrow[\text{Eq. of cats.}]{\sim} \mathrm{Vect}\,F_1 \underset{\mathrm{Vect}\,F_0}{\times} \mathrm{Vect}\,F_2$$

$$V \quad = \quad V_1 \cap V_2 \subseteq V_0$$

When does this happen?
It does if:

(1) Cartan's Lemma
(Factorization condition)
$\forall n \geq 1$, every matrix $A_0 \in GL_n(F_0)$
can be factored as
$$A_0 = A_1^{-1} A_2 , \quad A_i \in GL_n(F_i) \quad i = 1, 2$$

(2) Intersection:
$$F = F_1 \cap F_2 \subseteq F_0$$

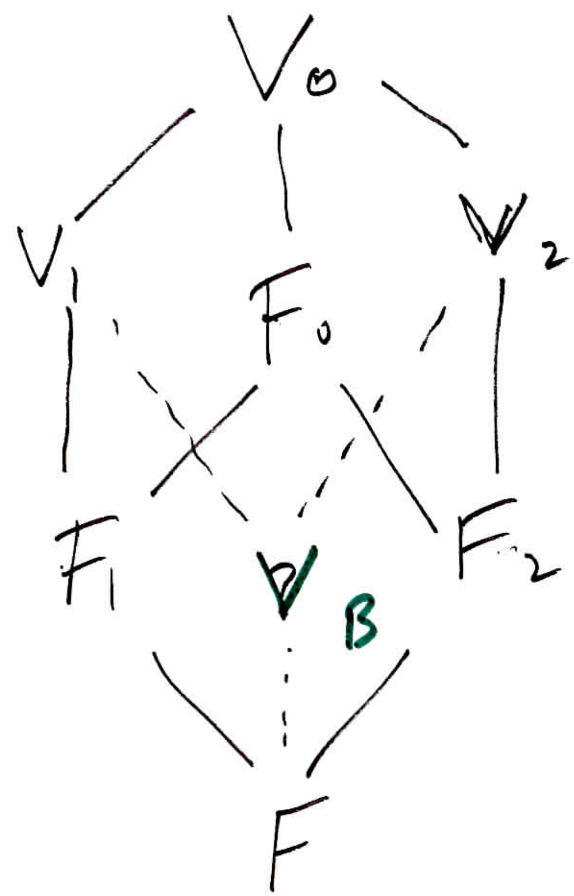
①, ② ⇒ patching:

$$B_1 \xrightarrow{A_0} B_2$$

$$B_1 A_0 = B_2$$

$$A_0 = A_1^{-1} A_2$$

$$B_1 A_1^{-1} \xleftarrow{A_1^{-1}} B_1 \qquad\qquad B_2 \xrightarrow{A_2^{-1}} B_2 A_2^{-1}$$



$$B := B_1 A_1^{-1} = B_2 A_2^{-1}$$

$$V = V_1 \cap V_2 \subseteq V_0$$

Other "structures"?

Ex. Fin dim assoc algs

A FDVS $/F$

mult $p : A \otimes_F A \longrightarrow A$

st assoc:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "p \otimes 1"] \arrow[d, "1 \otimes p"'] \arrow[dr, phantom, "\equiv"] & A \otimes A \arrow[d, "p"] \\
A \otimes A \arrow[r, "p"] & A
\end{tikzcd} \qquad (*)$$

Other structures?

— fin dim assoc alg w id

— " " comm alg

— " " sep " " ($\oplus$ of fin many fin. sep. fld exts)

— G-Galois F-algs (G fin gp)

($\oplus$ of fin many iso Galois fld exts)

(f.d. sep. F-alg A, w. a G-action st $F = A^G$ + $\dim_F A = |G|$.)

— csa

— diff'l + difference modules

Curve / cdvf K

e.g. $\mathbb{Q}_p$, $k((t))$

Function field e.g. $K(x) \longleftrightarrow \mathbb{P}^1_K$

$$\begin{array}{ccc} K(x) & & \mathbb{P}^1_K \\ \| & & \cap \\ F & & \mathbb{P}^1_T \end{array}$$

$$\begin{tikzcd} & F_0 \arrow[dl, dash] \arrow[dr, dash] & \\ F_1 \arrow[dr, dash] & & F_2 \arrow[dl, dash] \\ & F & \end{tikzcd}$$

$K = \mathrm{frac}\, T$ $\quad$ T cdvr, e.g. $\mathbb{Z}_p$, $k[[t]]$

(t), (p) maxl

res fld $= k$

$\hat{X} = \mathbb{P}^1_T$

$X = \mathbb{P}^1_k$

Spec T

generic fiber: $\mathbb{P}^1_K$