AWS 2012

Division Algebras and Patching

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$\mathbb{F}$ a field

Recall: A <u>central simple alg.</u> $/\mathbb{F}$ is a finite dimensional associative $\mathbb{F}$-algebra with no nontrivial two-sided ideals and has center $\mathbb{F}$

A <u>division algebra</u> $/\mathbb{F}$ is a csa $/\mathbb{F}$ in which every nonzero element has inverse.

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k$$

$$i^2 = j^2 = -1 \qquad ij = k = -ji$$

$\mathbb{H}$ has subfields $\cong \mathbb{R}, \mathbb{C}$

D division/F, $a \in D \setminus F$

$F \subsetneq F[a] \leq D$ $\quad\leadsto$ $F[a]$ subfield

Natural to study subfields.

Def: G a finite group

We say G is **admissible**/F if $\exists$

- a G-Galois field extension E/F
- an F-division algebra D containing E

s.th. $[E:F] = \deg_F(D) := \sqrt{\dim_F(D)}$

Ex: $G = \mathbb{Z}/2\mathbb{Z}$ complex conjugation on $\mathbb{C}/\mathbb{R}$

$\Rightarrow G$ admissible over $\mathbb{R}$.

Remarks:

1) If E/F is as in Def, then E is a maximal subfield of D

2) If G is admissible with division algebra D and ext. E/F, structure of D can be recovered from E and G.

Def: G finite group. A <u>G-crossed product</u> algebra A is defined by

- E/F finite G-Galois extension
- $A := \bigoplus_{\sigma \in G} E u_\sigma \qquad \sigma \in G$, $u_1 = 1$
- A 2-cocycle $c : G \times G \longrightarrow E^\times$

$$\sigma(c(\tau,\rho)) \cdot c(\sigma,\tau\rho) = c(\sigma\tau,\rho) \cdot c(\sigma,\tau)$$

for all $\sigma, \tau, \rho \in G$

- c normalized

$$c(1,\sigma) = 1 = c(\sigma,1) \quad \text{all } \sigma \in G$$

- multiplication defined by

$$u_\sigma \cdot b = \sigma(b)\cdot u_\sigma \quad \text{all } b \in E,\ \sigma \in G$$

$$u_\sigma \cdot u_\tau = c(\sigma,\tau)\, u_{\sigma\tau} \quad \text{all } \sigma,\tau \in G$$

<u>Lemma:</u> A G-crossed product algebra is a csa $/F$.

$$\mathbb{H} = (\mathbb{R} \oplus \mathbb{R}i) \oplus (\mathbb{R} \oplus \mathbb{R}i)j \qquad u_\sigma = j$$

$$\cup \qquad\qquad \langle \sigma \rangle = \mathbb{Z}/2\mathbb{Z}$$

$$u_\sigma \cdot a = j(\alpha + i\beta)$$

$$= (\alpha - i\beta) j$$

$$= \sigma(a) \cdot u_\sigma$$

Generally:

G admissible with division algebra D, E subfield

$\Rightarrow D$ is G-crossed product.

Cyclic algebras

G cyclic of order n, $E|F$ a G-Galois extension, $G = \langle \sigma \rangle$, $a \in F^\times$

$$0 \leq i, j \leq n-1$$

$$\xi_{\sigma,a}(\sigma^i, \sigma^j) = \begin{cases} 1 & i+j < n \\ a & i+j \geq n \end{cases}$$

Check: $\xi_{\sigma,a}$ is normalized 2-cocycle

<u>Def</u>: $A := (a, E|F, \sigma)$ is the crossed product algebra wrt. G and $\xi_{\sigma,a}$. Called <u>cyclic algebra</u>

$$A = \bigoplus_{i=0}^{n-1} E e^i$$

$$e \cdot b = \sigma(b) \cdot e \qquad b \in E$$

$$e^n = a$$

Theorem (Brauer, Hasse, Noether)

Every cyclic group is admissible over $\mathbb{Q}$

Theorem (Schacher, 1968)

If G is admissible over $\mathbb{Q}$ then all Sylow subgroups of G are metacyclic ($\leftarrow$ extension of cyclic by cyclic).

("Sylow-metacyclic")

Converse known for certain classes, e.g. solvable groups (Sonn)

conjectured in general (Schacher)

Theorem (HH, D. Krashen)

K complete discretely valued field with alg. closed residue field k

F a one variable function field over K

G a finite group, $\operatorname{char}(k) \nmid |G|$

Then:

G admissible over F $\Longleftrightarrow$ all Sylow subgroups of G are abelian metacyclic

<u>Here:</u> $F = \underbrace{k((t))}_{K}(x)$

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wts: G admissible $\Rightarrow$ every Sylow subgroup of G is abelian metacyclic.

<u>Recall:</u> A, B csa $\Rightarrow$ $A \otimes_F B$ csa

<u>Wedderburn:</u> Every csa is of the form $\mathrm{Mat}_n(D)$ some division algebra D.

$\rightsquigarrow$ define tensor product of division algebras

A, B division, $A \otimes B \cong \mathrm{Mat}_n(D)$, define: $A \cdot B = D$

$Br(F)$: set of division alg. $/F$ with this multiplication

Brauer group

$\alpha \in Br(F)$

per(α) : order in $Br(F)$, always finite.

F as before, Ω set of discrete valuations on F

$v \in \Omega$, let k_v denote residue field

$$Br(F)' := \{ \alpha \in Br(F) \mid (\mathrm{per}(\alpha), \mathrm{char}(k_v)) = 1 \text{ all } v \in \Omega \}$$

Known: $v \in \Omega \Rightarrow \exists$ homomorphism

$$\mathrm{ram}_v : Br(F)' \longrightarrow H^1(k_v, \mathbb{Q}/\mathbb{Z})$$

define $\mathrm{ram}_\Omega : \mathrm{Br}(F)' \longrightarrow \prod_{v \in \Omega} H^1(k_v, \mathbb{Q}/\mathbb{Z})$

Say that $\alpha \in \mathrm{Br}(F)'$ is <u>determined</u> by <u>ramification</u>

if $\mathrm{per}(\alpha) = \mathrm{per}(\mathrm{ram}_v(\alpha))$ for some $v \in \Omega$.

Colliot-Thélène, Ojanguren, Parimala: Fix prime p

$\exists\ \Omega$ s.th.

- ram_Ω is injective
- none of the residue fields k_v ($v \in \Omega$) has char. p

$\alpha \in Br(F) \quad \rightsquigarrow \quad \alpha = \alpha_p + (\alpha - \alpha_p)$

s.th. $\quad per(\alpha)_p = per(\alpha_p)$

$(per(\alpha - \alpha_p), p) = 1$

<u>Lemma:</u> $\alpha \in Br(F)$, Ω as above

$\Rightarrow$ α_p is determined by ramification.

Proof idea of „$\Rightarrow$" of Theorem:

1) fix prime p dividing $|G|$

2) let ℓ as above

D G- crossed product division alg. , max subfield E

$[D]_p$ is determined by ramification (Lemma)

w.r.t. that $v \in \mathcal{V}$

$\hat{E}/\hat{E}^P$ where P is p-Sylow subgroup.