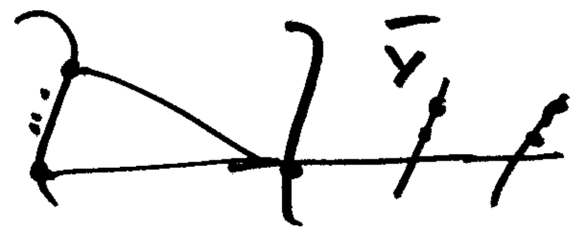
4th Lecture

Lifting results

$k \supseteq \mathcal{O} \to \bar{k} = \mathcal{O}/\mathfrak{m}$, G finite group

Given $f : Y \longrightarrow X$ G-Galois cover $/ k$

$\rightsquigarrow \bar{f} : \bar{Y} \longrightarrow \bar{X}$ stable red. of f



<u>Def</u>: A <u>stable G-map</u> is

$$\bar{f} : \bar{Y} \longrightarrow \bar{X} \quad , \text{ finite, flat}$$

$G \circlearrowright \bar{Y}$ $\qquad$ $\bar{Y}, \bar{X}$ semistable $/ \bar{k}$

(i) $\bar{Y}_i \subseteq \bar{Y}$, $\bar{X}_i := \bar{f}(\bar{Y}_i) \subseteq \bar{X}$ smooth

(ii) $y \in \bar{Y}^{\mathrm{sing}} \Rightarrow x = f(y) \in \bar{X}^{\mathrm{sing}}$

(iii) $I_i = \mathrm{Ker}(G_i \to \mathrm{Aut}_{\bar{k}}(\bar{Y}_i))$ is a p-group

(iv) $\bar{Y}_i \xrightarrow[\text{purely insep of degree } |I_i|]{} \bar{Z}_i \xrightarrow{G_i/I_i\text{-Galois}} \bar{X}_i$

We say that $\bar{f}$ <u>lifts</u> if it arises as stable red. of G-Galois cover

$$f : Y \to X$$

<u>Problem</u>: Characterize those $\bar{f}$ which lift!

<u>Ex:</u> Assume $I_i = 1$. Then:

$\bar{f}$ lifts $\Leftrightarrow$ $\bar{f}$ is <u>admissible</u>

(a) $\bar{f}$ is tamely ram over $\bar{X}^{smooth}$

(b) $y \in \bar{Y}^{sing} \rightsquigarrow G_y \cong \mathbb{Z}/m$, $(p,m)=1$

$$\hat{\mathcal{O}}_{\bar{Y},y} = \bar{k}[[\bar{u}_1, \bar{u}_2 \mid \bar{u}_1 \cdot \bar{u}_2 = 0]]$$

$$\tau^*(\bar{u}_1) = \zeta_m \bar{u}_1, \quad \tau^* = \zeta_m^{-1} \bar{u}_2$$

Goal. Generalize this to $|I_i| \in \{1, p\}$

$\leadsto$ Proof of Main Thm (1. lecture)

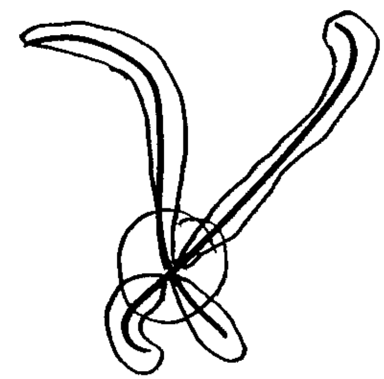
Idea: $|I_i| = p \leadsto$ deformation datum
$(\bar{Z}_i, \omega_i, \delta_i)$

Def: $(\bar{f}, (\bar{Z}_i, \omega_i, \delta_i))$ is admissible

if - - - -

$\bar{f}$ lifts $\Rightarrow$ (—) is admissible
$\overset{?}{\Leftarrow}$

"Formal patching"



$G \circlearrowright \overline{Y}$

- $\overline{Y}_i^{\circ} := \overline{Y}_i \setminus \overline{Y}^{\mathrm{sing}}$
  $= \mathrm{Spec}(\overline{B}_i)$

- $y \in \overline{Y}^{\mathrm{sing}} \rightarrow \overline{D}_y := \hat{\mathcal{O}}_{\overline{Y},y} \cong \bar{k}[[\bar{u}_1, \bar{u}_2 \mid \bar{u}_1 \cdot \bar{u}_2 = 0]]$

- $y \in \overline{Y}_i \rightarrow \overline{B}_{i,y} : \mathrm{Frac}(\overline{B}_i) \hookrightarrow \mathrm{Frac}(\overline{D}_y / \bar{\mathfrak{p}}_i)$

<u>Prop</u>: Assume there exist:

- G_i-equiv., $\mathcal{O}$-flat, complete lift R_i of $\overline{R}_i$
- $(\longrightarrow)$ lift D_v of $\overline{D}_v$
- emb. $\beta_{v,i} : \mathrm{Frac}(\hat{R}_i) \hookrightarrow \mathrm{Frac}(\widehat{D_{v,p_i}})$ lifting $\overline{\beta}_{v,i}$

Then $\overline{f}$ lifts!

<u>Proof:</u> construct <u>formal</u> $\mathcal{O}$-scheme $\hat{\mathcal{Y}}$ lifting $\bar{Y}$ + G-action s.t.

- $B_i \supset \Gamma(\bar{Y}_i^\circ, \mathcal{O}_{\hat{\mathcal{Y}}})$
- $D_y = \hat{G}_{\hat{\mathcal{Y}}, y}$
- $\beta_{y,i}$ „natural"

GET: $\hat{\mathcal{Y}}$ is formal compl. of $\mathcal{Y}$

$$\rightsquigarrow f : \mathcal{Y} \longrightarrow \mathcal{X} := \mathcal{Y}/G$$

$\blacksquare$

<u>Remark</u>: Gives much weaker result than "traditional" formal patching

- obtain lift $\mathfrak{X} = Y/G$, but we can't choose $\mathfrak{X}$
- works only for curves
- no deformation theory

How do we construct $(B_i, D_{Y_i}, \beta_{Y_i})$?

$\boxed{A}$ 1. Cases : $I_i = 1$

$\rightarrow \quad \bar{Y}_i^\circ = \mathrm{Spec}(\bar{B}_i) \longrightarrow \bar{X}_i^\circ = \mathrm{Spec}(\bar{A}_i)$

tamely ram. G_i – Galois cover

$\rightsquigarrow \quad \exists$ lift B_i / A_i (ess. unique)

2. Cases : $|I_i| = p$

Lemma : $\exists$ G_i – eq. lift B_i s.t.

$B_i / A_i = B_i^{G_i}$ gives rise $(\bar{Z}_i, \omega_i, \delta_i)$

Proof : deformation theory, group schemes

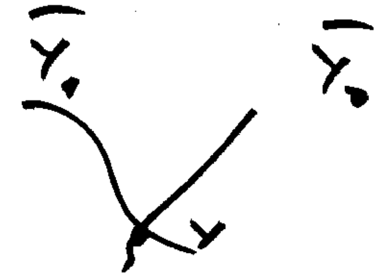
$\boxed{B}$ D_y <u>1. Case</u>: $G_y = \mathbb{Z}/m$, $(p,m)=1$

$\tau^* \bar{u}_1 = \zeta_m \bar{u}_1$, $\tau^* \bar{u}_2 = \zeta_m^{-1} \bar{u}_2$

$D_y := G[[u_1, u_2 \mid u_1 u_2 = \lambda]]$, $\lambda \in \mathfrak{m}\setminus\{0\}$

$\tau^* u_1 = \zeta_m u_2$, $\tau^* u_2 = \zeta_m^{-1} u_2$

<u>2. Case</u>: $G_y = \mathbb{Z}/p \times \mathbb{Z}/m$



For simpl.

- $m = 1$
- $|I_1| = |I_2| = p$

$h_i := \mathrm{ord}_y\, \omega_i - 1$. Then:

$$\boxed{h_1 + h_2 = 0}$$

One can write down explicitly

$$\mathcal{D}_y = \mathcal{O}[[u_1, u_2 \mid u_1 u_2 = \lambda]]$$

$$\circlearrowleft$$

$$G_y$$

<u>Cond:</u> $\delta_2 - \delta_1 = h_1 v(\lambda)$

C $\quad \beta_{\gamma,i}$?

Lemma: $K = \operatorname{Frac}(\widehat{\mathcal{O}[[t]](t^{-1})})$

Given two $G = \mathbb{Z}/p \rtimes \mathbb{Z}/m$ - ext.

$L_i / K \quad , \quad i = 1, 2.$

with the <u>same</u> data (r, h, δ)

then

$$\begin{array}{ccc} L_1 & \overset{G}{\xrightarrow{\sim}} & L_2 \\ | & & | \\ K & \xrightarrow{\sim} & K \end{array}$$