Group actions ... III:

Deformation data

Recall from yesterday:

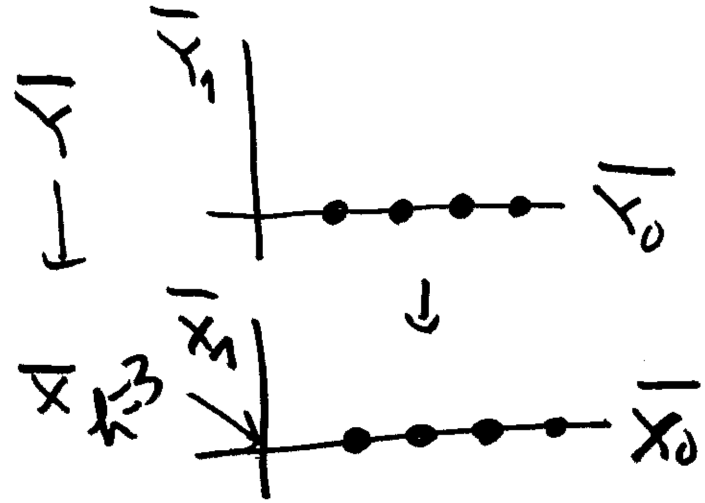
$p > 3$ $\quad Y \to X$, $\quad y^p = x^3 - px + 1$, $\quad (x,y) \mapsto x$, $\quad G = \mathbb{Z}/p\mathbb{Z} = \langle \sigma \rangle$, $\quad \sigma(x,y) = (x, \zeta_p y)$

in char 0.

G-stable red

$\bar{Y}_1 \to \bar{X}_1$ sep

$\bar{y}_1^p - \bar{y}_1 = \bar{x}_1^3$



$\bar{y}^p = \bar{x}^3 + 1$ insep

$$\omega_0 = \frac{3\bar{x}^2 \, d\bar{x}}{\bar{x}^3 + 1}$$

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{Y}' \text{ smooth} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X}' \end{array}$$

contract $\bar{Y}_0$, $\bar{X}_0$

Situation : $G = \mathbb{Z}/p\mathbb{Z}$

$\mathcal{Y} \longrightarrow \mathcal{X}$ G-stable

let $\bar{Y}_0$ irred comp of $\bar{Y}$

st $I(\bar{Y}_0) \neq \{1\}$.

Complete $k(\mathcal{Y})$ $(k(\mathcal{X}))$ w/t

val. corr. $\bar{Y}_0$ $(\bar{X}_0)$.

$\rightsquigarrow \hat{L}/\hat{K}$ $\quad$ c/f. resfield ext is insep

$\hat{L} = \hat{K}[y]/(y^p - g(x))$ $\quad$ Kummer

III-2

Classification :

$\boxed{A}$ Multiplicative case $\quad \bar{g} \notin \bar{K}^p$

$\rightsquigarrow \quad \bar{L} = \bar{K}[\bar{y}]/(\bar{y}^p - \bar{g})$

Associate : • depth $\delta = \frac{p}{p-1}$

• differential Swan conductor :

$$\omega = \frac{d\bar{g}}{\bar{g}}$$

<u>Exa</u> $\omega_0 = \frac{d\bar{g}}{\bar{g}} = \frac{3\bar{x}^2 d\bar{x}}{\bar{x}^3+1}$

$\boxed{B}$ <u>Additive case</u> $\bar{g} \in \bar{K}^p$

$\rightsquigarrow$ $\bar{y}^p = \bar{g}$ reducible

write $g(x) = z^p[1+\mu^p u]$, $\mu \in \mathfrak{m}$

$z, u \in \mathcal{O}_E^\times$ def $y = z(1+\mu w)$

$\rightsquigarrow$ $\bar{w}^p = \bar{u} \notin \bar{K}^p$ (may assume)

depth $\delta = \frac{p}{p-1} - v(\mu)$

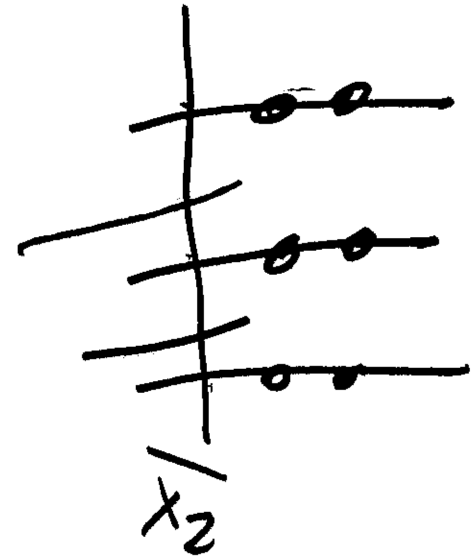
$\rightsquigarrow$ diff Swan: $\omega = d\bar{u}$.

$\{ xa \} \quad y^2 = x(x^4 - 1)$

$\mathbb{Z}/2\mathbb{Z}$

X

$\overline{X}$



$\overline{X_2}: \quad x = \sqrt{2}\, x_2 + 1$

$y = 2 y_2$

$\rightsquigarrow \quad \overline{y_2}^2 = \overline{x_2}^2 (\overline{x_2} + 1)^2 \in \bar{k}^2$

III.5

Hurwitz tree

- tree of $\mathbb{P}^1$'s ($\overset{?}{=}$ $\bar{X}$)
- marked pts ($\overset{?}{=}$ sp. of ram pts)
- Decomp & Inertia gps
- ram of sing pts / marked
- insep compt: (δ, ω)

+ comp. cond.

Construction of defo. data

mult. case: $\omega = \frac{d\bar{s}}{\bar{s}}$ log. diff form

Add case: $\omega = d\bar{u}$ exact

<u>Lem</u> Let $h > 0$ prime to p $G = \mathbb{Z}/p\mathbb{Z}$

$\exists$ log. diff. form ω on $\mathbb{P}^1$

$h+1$ simple poles & single zero order $h-1$

Pf $\bar{x}_i = \zeta_n^i \quad i=1,\dots,n \;, \bar{x}_0 = 0$

$$\bar{g} = \bar{x}^{-n} \prod_{i=1}^{n} (\bar{x} - \bar{x}_i)$$

$$\frac{d\bar{g}}{\bar{g}} = \left[-\frac{n}{\bar{x}} + \sum_{i=1}^{n} \frac{1}{\bar{x} - \bar{x}_i} \right] d\bar{x}$$

$$= \frac{n}{\bar{x}^{n+1} - \bar{x}} \, d\bar{x}$$

$\square$

Hurwitz trees for $G = \mathbb{F}_p \rtimes_\chi \mathbb{Z}/m\mathbb{Z}$.

where $\chi : \mathbb{Z}/m\mathbb{Z} \hookrightarrow \mathbb{F}_p^\times$. $(m \mid p-$

BO vanishes $\Leftrightarrow$ $h \equiv -1 \pmod m$

ass $h \equiv -1 \pmod m$ $\qquad n = \left(\frac{p-1}{m}\right)$

$h+1 = m[\alpha + p\beta]$

$\boxed{\text{Case A}}$: $1+n < \alpha < p$, $\beta \neq 0$

$\overline{Y}_1$

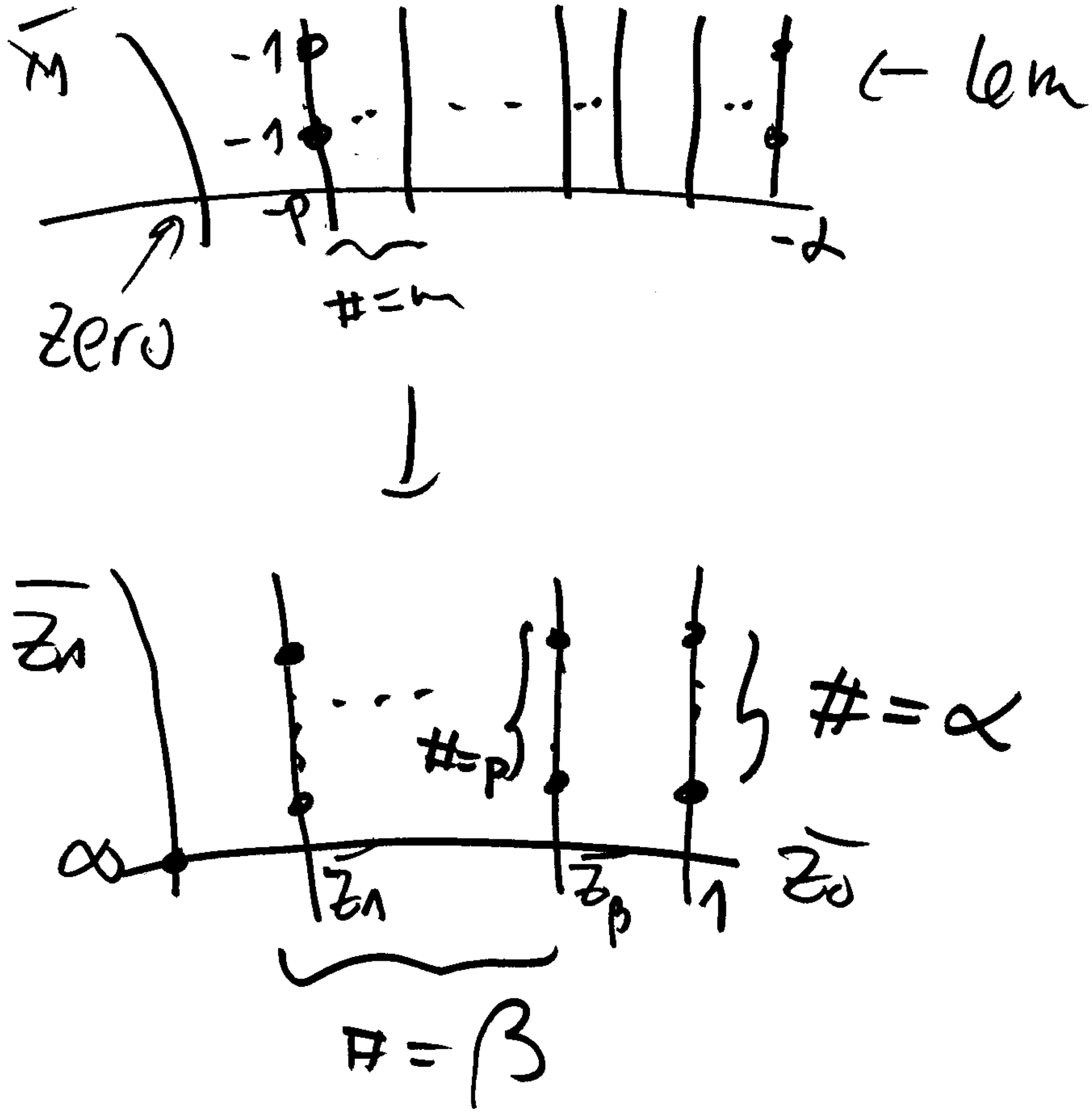
$\mathbb{Z}/p\mathbb{Z}$

$\overline{X}_1 = \mathbb{P}^1$

$\mathbb{Z}/m\mathbb{Z}$

$\overline{Z}_1 = \mathbb{P}^1$

sep



Once we have chosen ω's on tails $\leadsto$ ω_0 determined.

$$\omega_0 = \frac{d\bar{x}_0}{(\bar{x}_0^m - 1)^\alpha \prod_i (\bar{x}_0^m - z_i)^\nu} \overset{!}{=} d\alpha^-$$

HJS : ω_0 exact. OK : choice of α

HTS $\frac{d\bar{x}_0}{(\bar{x}_0^m-1)^\alpha}$ exact.

$$= \sum_i \left(\frac{c_{\alpha,i}}{(\bar{x}_0 - \zeta_m^i)^\alpha} + \dots + \cancel{\frac{c_{1,i}}{(\bar{x} - \zeta_m^i)}} \right)$$

III-11

Show : all residues are zero.