AWS 12

Group actions on curves and the local lifting problem

S. Wewers + I. Bouw

1st Lecture

- p prime #

- $\mathcal{O}$ complete dvr

  - $k = \mathrm{Frac}(\mathcal{O})$ char 0
  - $\bar{k} = \mathcal{O}/m$ char p, $\bar{k} = \bar{k}^{ac}$
  - $v : k^\times \to \mathbb{Q}$, $v(p) = 1$

Convention k suff. large

LLP

$\bar{A} = \bar{k}[[t]]$, $G \subseteq \mathrm{Aut}_{\bar{k}}(\bar{A})$. Does $(\bar{A}, G)$ lift?

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, dashed, "?"'] & \mathrm{Aut}_{\bar{k}}(\bar{A}) \\ & \mathrm{Aut}_{\mathcal{O}}(A) \arrow[u] \end{tikzcd} \qquad A = \mathcal{O}[[t]]$$

Thm: $(\bar{Y}, G)$ lifts $\iff \forall y \in |\bar{Y}|$: $(\hat{\mathcal{O}}_{\bar{Y}, y}, G_y)$ lifts

<u>LP</u>: $\bar{Y}/\bar{k}$ smooth proj curve, $G \subseteq \mathrm{Aut}_{\bar{k}}(\bar{Y})$

Does $(\bar{Y}, G)$ lift?

$\exists\ \mathcal{Y} \to \mathrm{Spec}(\mathcal{O})$ proper smooth s.t.

- $\mathcal{Y} \otimes \bar{k} = \bar{Y}$
- $$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, dashed, "?"'] & \mathrm{Aut}_{\bar{k}}(\bar{Y}) \\ & \mathrm{Aut}_{\mathcal{O}}(\mathcal{Y}) \arrow[u] \end{tikzcd}$$

<u>Rem:</u> Let $G \subseteq \mathrm{Aut}_{\bar{k}}(\bar{k}((t)))$ finite. Then

$$G = P \rtimes C,$$

where
- P p group
- $C \cong \mathbb{Z}/m$, $(m, p) = 1$

Moreover, $\tau(t') = \zeta_m \cdot t'$, $C = \langle \tau \rangle$

For the rest of this talk:

- $G = P \rtimes_\chi C$
- $P \cong \mathbb{Z}/p\mathbb{Z}$
- $\chi : C \longrightarrow \mathbb{F}_p^\times$

$$\tau \sigma \tau^{-1} = \sigma^{\chi(\tau)}$$

χ nontrivial $\Leftrightarrow$ G not abelian

Let $(\bar{A}, G)$ be a local action

Katz - Gabber : $\exists$ G-Galois

cover

$$\bar{f} : \bar{Y} \xrightarrow{G} \bar{X} = \mathbb{P}^1_{\bar{k}}$$

- étale over $\bar{X} - \{0, \infty\}$
- $\bar{f}$ is tamely ram. over 0, inertia group C
- $\bar{f}$ is purely ram. over ∞,

$$\bar{A} \underset{G}{\cong} \hat{\mathcal{O}}_{\bar{Y},\infty}$$

$$\begin{tikzcd} & \mathbb{P}^1_{\mathbb{F}_q} \arrow[d, equal] & \\ \bar{Y} \arrow[r, "p"] & \bar{Z} := \bar{Y}/p \arrow[r, "c"] & \bar{X} = \mathbb{P}^1_{\mathbb{F}_q} \end{tikzcd}$$

$$\bar{z}^m = \bar{x}$$

$$\bar{y}^p - \bar{y} = \bar{z}^h + \dots + c_0 \quad , \quad (h,p) = 1$$

$$g(\bar{Y}) = \frac{(p-1)(h-1)}{2}$$

Assume $(\bar{Y}, G)$ lifts:

$$f : \mathcal{Y} \xrightarrow{p} \mathcal{Z} = \mathcal{Y}/p \xrightarrow{c} \mathcal{X} = \mathbb{P}^1_{\mathcal{O}}$$

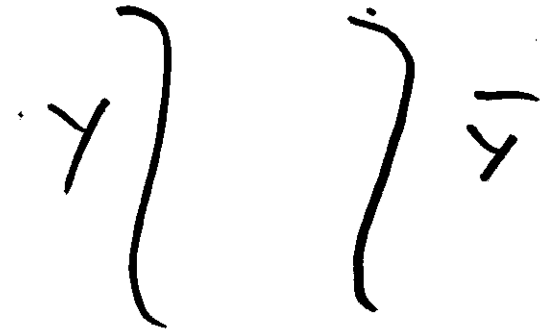
$$y^p = u \in \mathcal{K}(z) \qquad\qquad z^m = x$$

$$u = \prod_i (z - \alpha_i)^{a_i}, \quad (a_i, p) = 1, \quad \sum a_i = 0$$

$$v(\alpha_i) < 0$$

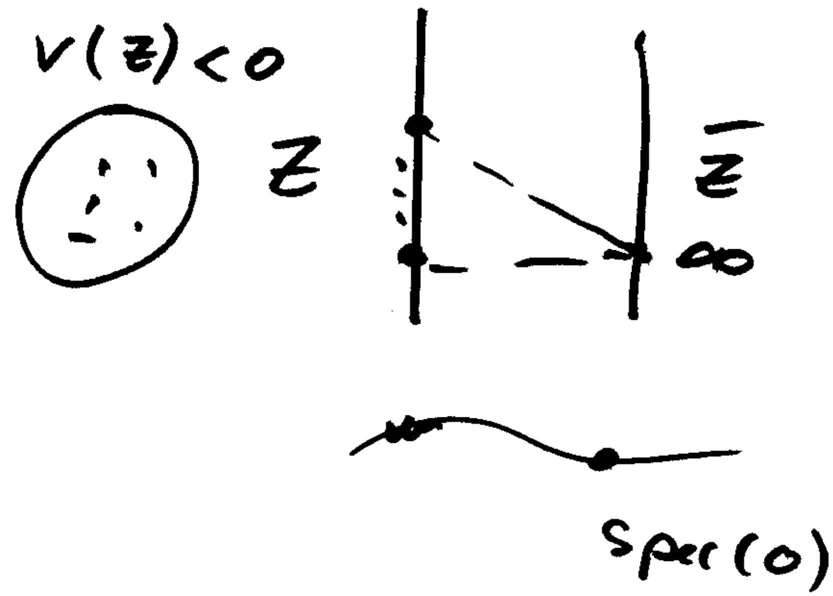
Assumption

$v(\alpha_i) = v(\alpha_i - \alpha_j) = v(\lambda^{-1})$,



$\lambda \in \mathfrak{m}$

$z = \lambda^{-1} z_1$



$$u = \prod_i (\lambda^{-1} z_1 - \alpha_i)^{a_i}$$

$$= \prod_i (z_1 - \underbrace{\lambda \alpha_i}_{v(-)=0})^{a_i}$$

$$\rightsquigarrow \bar{u} = \prod_i (\bar{z}_1 - \bar{\alpha}_i)^{a_i}, \quad \bar{\alpha}_i \neq \bar{\alpha}_j$$

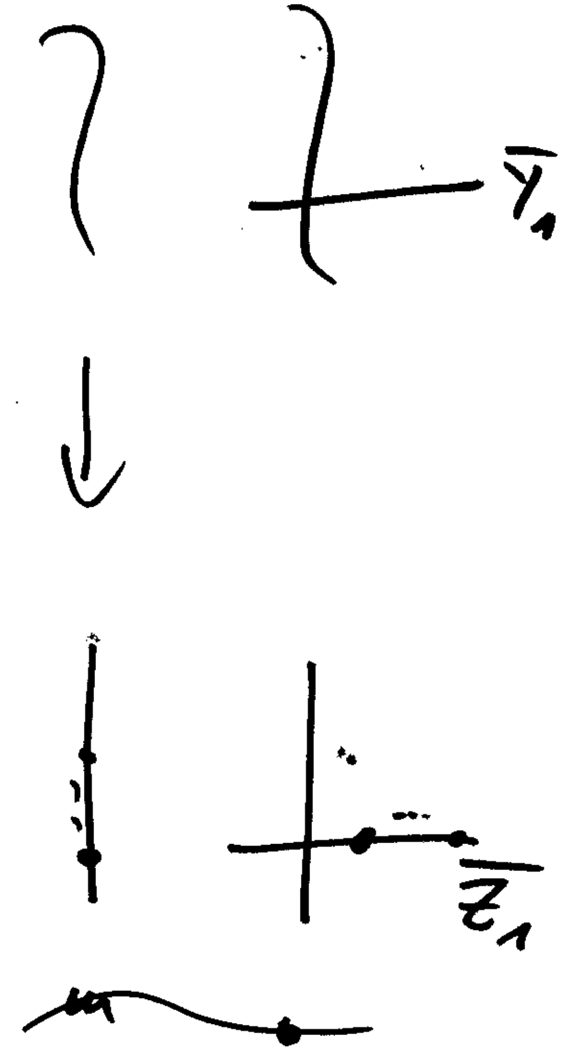
$$\begin{tikzcd} \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathcal{Z}' \arrow[r] & \mathcal{Z} \end{tikzcd}$$

$\overline{Y}_1 \longrightarrow \overline{Z}_1$ finite

purely insep. map between smooth curves of $g = 0$,

$$\bar{y}^p = \bar{u}$$

$$\omega := \frac{d\bar{u}}{\bar{u}} = \sum_{i=1}^{h+1} \frac{a_i \, d\bar{z}_1}{(\bar{z}_1 - x_i)}$$



Prop: (i) $\tau^* \omega = \chi(\tau) \cdot \omega \ \forall \tau \in C$

(ii) ω has a <u>unique</u> zero at $\bar{z}_1 = \infty$ of order $h-1$

Cor: (i) χ is injective $(\leadsto m \mid p-1)$

(ii) $m \mid h+1$

Rem. This gives a necessary cond. for lift. of $(\bar{A}, G)$!

Thm (B.-W.-Zapponi)

$(\bar{A}, G)$ lifts $\Leftrightarrow$ (i) + (ii) holds