PERIODS AND SPECIAL VALUES OF L-FUNCTIONS

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Contents

Introduction

These are lecture notes for my course at the Arizona Winter School. The style is somewhat informal and I have aimed to convey the key ideas without many technical details. [1] The theme of these lectures will be *period relations* between modular forms. Two reasons (amongst others) why one might be interested in periods are:

- Periods occur in various special value conjectures like the Birch-Swinnerton-Dyer, and its generalization, the Bloch-Kato conjecture.

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[1] The notes have not been seriously proofread as yet, and are also somewhat incomplete at this point, especially in Sections 7 and 8. On the other hand, the material relevant to the student projects is all there.

- Period relations are often a manifestation of the Hodge/Tate conjectures. For example, the Hodge conjecture predicts that an isomorphism of rational Hodge structures - which can be viewed as a collection of period relations - comes from an algebraic cycle.

The theory of Shimura varieties and automorphic forms provides a fascinating testing ground for the conjectures mentioned above, since the functoriality conjectures of Langlands often predict relations between automorphic forms on different Shimura varieties. In many instances, one then expects an algebraic cycle to be lurking around. In no case, beyond that of a product of two (Shimura) curves, has the existence of such a cycle ever been demonstrated in general. In that case too, there is no canonical construction of such a cycle; rather its existence follows from a deep general theorem of Faltings that works for any product of curves. One might optimistically hope that a study of period relations may lead ultimately to such a construction or a better understanding of the Tate conjecture in this setting.

The study of period relations for automorphic forms was pioneered by Shimura, who showed in many instances the existence of relations up to factors in $\overline{\mathbf{Q}}^{\times}$, and made a general conjecture relating relating periods on Hilbert modular varieties and their compact analogs, the so-called quaternionic modular varieties. There is a somewhat weaker conjecture, that relates quadratic periods (which as the name suggests are a product of two periods) which may be interpreted (up to algebraic factors) as Petersson inner products. It is this weaker conjecture that we will focus on in these notes. In fact, the conjecture itself was proven by Michael Harris [8] under a certain technical condition that we will describe later. [2] However, we shall be interested in a more precise formulation of this conjecture that works not just up to algebraic factors but up to factors that are units (or at least p-adic units for a fixed prime p).

The motivation for this more precise formulation comes from another venerable tradition in the theory of modular forms, namely the study of congruences, which in its modern formulation, due to Hida, Ribet and ultimately in the work of Wiles and Taylor-Wiles has been a key ingredient in the some of the most important developments in algebraic number theory in recent years. We will begin by discussing this motivation.

1. Modular forms, congruences and the adjoint L-function

We summarize various well known facts about classical modular forms. Let N be a positive integer, $k \geq 2$ an integer and $S_k(\Gamma_0(N))$ the space of cusp forms of weight k on $\Gamma_0(N)$. This space admits an action by a Hecke algebra $\mathbb{T}_{k,N}$, generated by Hecke operators T_p for $p \nmid N$. We let $f \in S_k(\Gamma_0(N))$ be a newform in $S_k(\Gamma_0(N))$ i.e. it is an eigenform for $\mathbb{T}_{k,N}$ and the corresponding system of eigenvalues (for $p \nmid N$) does not appear in $S_k(\Gamma_0(M))$ for any M

[2] And that can now be removed as a consequence of the ideas described in Sec. 8.

strictly dividing N. We will assume that f is normalized i.e. has a q-expansion

$$f(z) = \sum_{n\geq 1} a_n q^n, \qquad \text{where} \quad q := e^{2\pi i n z} \text{ and } a_1 = 1.$$

The coefficients $a_p = a_p(f)$ for p prime are related to the Hecke eigenvalues by

$$T_p f = a_p f, \qquad \text{for } p \nmid N.$$

Let K_f denote the subfield of $\mathbf{C}$ generated by the a_n. Then K_f is a totally real number field. Let K be any number field containing K_f, $\mathcal{O}$ the ring of integers of K. Let ℓ be a rational prime and fix an embedding λ of $\overline{\mathbf{Q}}$ in $\overline{\mathbf{Q}}_\ell$. We will denote by K_λ the completion of K at λ, $\mathcal{O}_\lambda$ its ring of integers and $\mathbf{F}_\lambda$ its residue field. The following fundamental result is due to Shimura and Deligne.

Theorem 1.1. *There is associated to f a continuous representation (well defined up to isomorphism)*

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathbf{GL}_2(K_{f,\lambda})$$

satisfying

(1) *$\rho_{f,\lambda}$ is unramified outside ℓN.*
(2) *For any $p \nmid \ell N$, the characteristic polynomial of* Frob_p *acting on $\rho_{f,\lambda}$ is $X^2 - a_p X + p^{k-1}$.*

There is also an associated mod λ representation. Namely, since the Galois group is compact, one can conjugate $\rho_{f,\lambda}$ to take values in $\mathbf{GL}_2(\mathcal{O}_\lambda)$ and then reduce mod λ. The semisimplication of this reduction can be shown to be independent of choices up to isomorphism, and the associated representation into $\mathbf{GL}_2(\mathbf{F}_\lambda)$ will be denoted $\bar{\rho}_{f,\lambda}$. Note that the condition (2) above implies that if $\rho_{f,\lambda} \simeq \rho_{g,\lambda}$, then $f = g$. On the other hand, it is certainly possible for two different newforms f and g (of possiby different level) to have the same associated mod λ-representations. Clearly this happens exactly when for all but finitely many primes p, we have

$$a_p(f) \equiv a_p(g) \mod \lambda.$$

We say in this case that f and g are congruent mod λ.

The systematic study of such congruences began with the work of Hida and Ribet in the early 80's and culminated in the work of Wiles and Taylor-Wiles on the Taniyama-Shimura conjecture. We will describe some of these results since they will provide a context for the main theme of this lecture series: (for simplicity, we will assume that λ is prime to all the levels that occur; we also ignore powers of π and other constants that are not key to the discussion.)

- Hida ([12], [13]) showed that there is a canonical period (which we will call Ω_f) such that a prime λ divides the ratio

$$\delta_f := \frac{L(k, \mathrm{Sym}^2 f)}{\Omega_f}$$

if and only if f satisfies a congruence mod λ with another newform g in $S_k(\Gamma_0(N))$ (of level possibly smaller than N.) These days, it is customary to replace the symmetric square L-function by the adjoint, which is just given by a shift. The L-value of interest is then $L(1, \mathrm{ad}^0 f)$, where the general L-factor is

$$L_p(s, \mathrm{ad}^0 f) = (1 - \frac{\alpha_p}{\beta_p} p^{-s})(1 - \frac{\beta_p}{\alpha_p} p^{-s})(1 - p^{-s}),$$

with α_p and β_p being the Hecke eigenvalues of f at p.

- Ribet ([24],[26]) proved a level-raising result: namely, f is congruent mod λ to a newform g of level dividing Np and that is new at p if and only if λ divides the value of Euler factor $L_p(s, \mathrm{ad}^0 f)$ at $s = 1$.
- Ribet ([27]) proved also a level-lowering result: namely, if $p \mid N$, then f is congruent to a newform g of level dividing N/p if and only if $\bar{\rho}_{f,\lambda}$ is unramified at p. In the case of semistable elliptic curves for instance, this is equivalent to saying that λ divides c_p, the order of the component group of the Neron model of E at p.
- Finally, Wiles [41] defined a very precise measure of congruences (the η-invariant η_f). The articles [41] and [34] show firstly that

$$v_\lambda(\eta_f) = v_\lambda(\delta_f), \tag{1}$$

and secondly that $\#(\mathcal{O}_\lambda/\eta_f)$ is the size of a certain Selmer group $H^1_\Sigma(\mathbf{Q}, \mathrm{ad}^0 \rho_{f,\lambda} \otimes K_\lambda/\mathcal{O}_\lambda)$. This is essentially the Bloch-Kato conjecture for the adjoint L-value at 1.

A word about the relation (1) above. The eta-invariant is defined by looking at a suitable localization $\mathbb{T}_\Sigma$ of the Hecke algebra, which is a reduced $\mathcal{O}_\lambda$-algebra equipped with a map

$$\pi : \mathbb{T}_\Sigma \to \mathcal{O}_\lambda$$

corresponding to the Hecke action on f. (Here we have possibly extended scalars so that $\mathcal{O}_\lambda$ contains all Hecke eigenvalues of all forms in a certain finite set consisting of forms of controlled levle and congruent to f mod λ.) Wiles then defines

$$(\eta_f) = \pi(\mathrm{Ann}(\ker \pi)).$$

The following lemma (copied from [4]: see Lemma 4.17) is the key ingredient used to show (1):

Lemma 1.2. *Suppose that there is a $\mathbb{T}_\Sigma$-module $\mathcal{L}$ satisfying the following properties:*

(1) *$\mathcal{L}$ is finitely generated and free over $\mathcal{O}_\lambda$.*
(2) *$\mathcal{L} \otimes_{\mathcal{O}_\lambda} K_\lambda$ is free of rank d over $\mathbb{T}_\Sigma \otimes_{\mathcal{O}_\lambda} K_\lambda$.*
(3) *$\mathcal{L}$ is equipped with a perfect $\mathcal{O}_\lambda$-bilinear pairing*

$$\langle , \rangle : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_\lambda$$

satisfying $\langle Tx, y \rangle = \langle x, Ty \rangle$.

Then, letting $\mathfrak{p} := \ker(\pi)$*, the module* $\mathcal{L}[\mathfrak{p}]$ *is free of rank* d *over* $\mathcal{O}_\lambda$ *and for any basis* $(x_1, \ldots, x_d)$ *of* $\mathcal{L}[\mathfrak{p}]$*, the relation*

$$\eta_f^d \subseteq \mathcal{O}_\lambda \cdot \det(\langle x_i, x_j \rangle)$$

holds. Further, if $\mathcal{L}$ *is free as a* $\mathbb{T}_\Sigma$*-module, then we have equality above.*

In the case of interest above, the module $\mathcal{L}$ is constructed from the H^1 of a modular curve X and can be shown to be free of rank 2 over $\mathbb{T}_\Sigma$. The pairing is given by tweaking the cup-product and is skew-symmetric. Taking a basis x, y, for $\mathcal{L}[\mathfrak{p}]$ that is K-rational, and translating to de Rham cohomology, one finds that

$$(\eta_f) = (\langle x, y \rangle) = \frac{\langle f, f \rangle}{\Omega_f} \tag{2}$$

where Ω_f is the determinant of the change of basis matrix from (x, y) to $(\omega_f, \overline{\omega}_{f^\rho})$ with ω_f the one-form on X associated to f and $\langle f, f \rangle$ denotes the Petersson inner product:

$$\langle f, f \rangle = \frac{1}{\mathrm{vol}(\mathcal{H}/\Gamma)} \int_{\Gamma \backslash \mathcal{H}} f(z)\overline{f(z)} dx dy.$$

Finally, one uses a formula of Rankin and Shimura that shows:

$$\langle f, f \rangle = L(1, \mathrm{ad}^0 f).$$

Consequently, we get the following:

BASIC PRINCIPLE:

$$\frac{\text{Petersson inner product}}{\text{canonical period}} = \text{congruence number}.$$

We shall shortly explore how this generalizes to other contexts. For the moment, let us note that in the classical situation, one can view cuspforms in two other ways:

(i) As functions on the group $\mathbf{GL}_2(\mathbf{Q}) \setminus \mathbf{GL}_2(\mathbb{A})$: Let M be the order

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) : c \equiv 0 \mod N \right\},$$

and let U be the open compact subgroup of $\mathbf{GL}_2(\mathbb{A}_f)$ defined by

$$U := \prod_q U_q, \qquad U_q := (M \otimes \mathbf{Z}_q)^\times.$$

Then, by strong approximation, we can write any $g \in \mathbf{GL}_2(\mathbb{A})$ as

$$g = g_{\mathbf{Q}} \cdot (g_U \cdot g_\infty)$$

where $g_{\mathbf{Q}} \in \mathbf{GL}_2(\mathbf{Q})$, $g_U \in U$ and $g_\infty \in \mathbf{GL}_2(\mathbf{R})^+$. Finally, we pick a base point $i \in \mathcal{H}$ and define

$$F(g) = f(g_\infty(i))j(g_\infty, i)^{-k}$$

where $j(\gamma, z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathbf{R})^+$ is the automorphy factor

$$j(\gamma, z) := (\det \gamma)^{-1/2}(cz + d).$$

The function F lives in $L^2(\mathbf{GL}_2(\mathbf{Q}) \setminus \mathbf{GL}_2(\mathbb{A}))$, and if f is a Hecke eigenform, generates an *automorphic representation* of $\mathbf{GL}_2(\mathbb{A})$, which we will call π_f.

(ii) As sections of a line bundle on the modular curve $X_0(N) = \overline{\mathcal{H}/\Gamma_0(N)}$. i.e.

$$(2\pi i)^{k/2} f(z) dz^{\otimes k/2}$$

is Γ-invariant and gives a section of $\Omega^{k/2}$ on $Y_0(N)$ which extends to $X_0(N)$.

2. Quaternion algebras and the Jacquet-Langlands correspondence

One would like to extend the picture from the previous section to other algebraic groups. The next simplest case is the multiplicative group of a quaternion algebra. We will start by discussing the case of *indefinite* quaternion algebras since that is structurally similar to the previous section. Henceforth, we assume that N is square-free since that makes things a little easier to describe.

Let B be an indefinite quaternion algebra over $\mathbf{Q}$. Thus B is ramified at an even number of finite primes, the discriminant of B being the product of these. We will assume that the discriminant of B is a divisor N^- of N and write $N = N^+ \cdot N^-$.

Let us fix an isomorphism

$$\iota_\infty : B \otimes \mathbf{R} \simeq M_2(\mathbf{R}),$$

as well as isomorphisms

$$\iota_q : B \otimes \mathbf{Q} \simeq M_2(\mathbf{Q}_q),$$

for all q that are split in B. Then there is a unique order M_B in B satisfying the following conditions:

(i) $\iota_q(M_B) = M \otimes \mathbf{Z}_q$ for $q \nmid N^-$.

(ii) For $q \mid N^-$, $\iota_q(M_B)$ is the unique maximal order in $B_q := B \otimes \mathbf{Q}_q$.

M_B is an Eichler order i.e. an intersection of two maximal orders. Let Γ_B be the (multiplicative) group of elements with reduced norm 1 in M_B. It is analogous to $\Gamma_0(N)$, and via ι_∞ one can consider Γ_B as a discrete subgroup of $\mathbf{SL}_2(\mathbf{R})$. As in the classical case, one can study modular forms of weight k with respect to Γ. One has analogously Hecke operators T_p (say for $p \nmid N$) and we can consider eigenforms for the Hecke algebra generated by these. Again, we may think of such modular forms in two different ways:

(i) As automorphic forms on the group $B^\times \setminus B_{\mathbb{A}}^\times$.

(ii) As sections of the line bundle $\Omega^{k/2}$ on the Shimura curve $X_B = \mathcal{H}/\Gamma$.

Remark 2.1. The key difference between this case and the case of usual modular forms is that X_B is compact if B is a division algebra. One may imagine that this makes things easier to study, while in fact the opposite is true for arithmetic questions, since q-expansions are not available. For example, it is not clear how to normalize eigenforms or define notions of rationality or integrality. It is also very hard to do computations on such curves: one of the projects in this course will be to attempt to develop methods to compute effectively with modular forms in this setting.

What gets us going is that the curve X_B is not just defined over $\mathbf{C}$, but has a *canonical model* over $\mathbf{Q}$, defined by Shimura. This model is characterized by requiring that if K is an imaginary quadratic field embedded in B, then the (unique) fixed point on $\mathcal{H}$ of $\iota_\infty(K^\times) \hookrightarrow \mathbf{GL}_2(\mathbf{R})^+$ is algebraic and further $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on such points in a prescribed manner. In fact, X_B can be shown to have good reduction outside of N i.e. it has a smooth model $\mathcal{X}_B$ over $\mathrm{Spec}\,\mathbf{Z}[\frac{1}{N}]$. This makes it possible to define rational and λ-integral (for $\lambda \nmid N$) structures on the space of modular forms of weight k by requiring that the corresponding sections of $\Omega^{k/2}$ be rational or λ-integral. Let g be an eigenform that is normalized in this way to be rational over the field of Hecke eigenvalues and λ-integral. Then exactly as in the previous section, one can define a canonical period Ω_g associated to g and an η-invariant, η_g, and show that

$$(\eta_g) = \frac{\langle g, g\rangle}{\Omega_g}, \tag{3}$$

provided we have the requisite freeness results for the homology of X_B over the Hecke algebra (suitably localized). Such freeness results are known to hold in weight 2 under certain conditions (see Helm[11] and Ribet[28]) but not in the same generality as is known for modular curves. In general, we at least get $(\eta_g) \subseteq (\delta_g) := \frac{\langle g,g\rangle}{\Omega_g}$.

We would like to compare (2) with (3) when f and g are related by the Jacquet-Langlands correspondence. Recall that the Jacquet-Langlands correspondence says that given such a g, then there is an associated form f that is uniquely determined by requiring that the eigenvalues of the Hecke operators T_p (for $p \nmid N$) acting on g and f are equal. Conversely, because of our assumptions on f being new of level N, there is an associated g on B that corresponds to f. (In general, if we only assumed that f had level N' dividing N, there is an associated g exactly when N^- divides N'.) Since the Hecke algebra for B is a quotient of the one for $\mathbf{GL}_2$, one gets a containment:

$$(\eta_f) \subseteq (\eta_g).$$

One further expects that $\Omega_f = \Omega_g$ at least up to primes that are Eisenstein i.e. $\Omega_f/\Omega_g \in K_f$ and if $\bar\rho_{f,\lambda}$ is irreducible, then this ratio is a λ-adic unit. This is hard to prove in general, but suggests that the ratio

$$\frac{\langle f, f\rangle}{\langle g, g\rangle} \sim_\lambda \frac{\eta_f}{\eta_g}$$

should "count" congruences between f and other newforms of level dividing N but not divisible by N^-.

Another case one might consider is that of definite quaternion algebras. In this case, the associated "Shimura varieties" are just finite sets of points i.e. the geometry is very simple, so all the interesting information is in the arithmetic of the quaternion algebra. Suppose that B is a definite quaternion algebra, of discriminant $N^-\infty$ (so that N^- is divisible by an odd number of primes). Again, we let M_B denote the unique order in B satisfying (i) and (ii). The associated Shimura variety is just the set of right M_B-ideal classes:

$$\mathrm{Cl}(M_B) := B^\times \setminus \hat{B}^\times / \hat{M}_B^\times.$$

Let $\{x_1, \ldots, x_s\}$ be a set of coset representatives for the elements in $\mathrm{Cl}(M_B)$, and let $\Gamma_i := x_i \hat{M}_B^\times x_i^{-1} \cap B^\times$. Then each Γ_i is a finite group, whose size we denote by w_i. The Jacquet-Langlands correspondence associates to f of weight 2 a form g on $B(\mathbf{Q}) \setminus B(\mathbb{A})$ which is a complex valued function well defined up to scaling, right invariant by $B_\infty^\times$ and $\hat{M}_B^\times$ and left invariant by $B^\times$ i.e. a complex valued function on $\mathrm{Cl}(M_B)$. [3] Let $S_2(B)$ denote the space of such functions. The Petersson inner product (defined adelically using the Tamagawa measure) can be shown to translate to the following inner product:

$$\langle g_1, g_2 \rangle = \sum_i \frac{1}{w_i} g_1(x_i) \overline{g_2(x_i)}.$$

Let $\mathcal{M}$ denote the module of $\mathcal{O}$-valued functions on $\mathrm{Cl}(M_B)$. We can normalize g so that it lies in $\mathcal{M}$ and is primitive. There is as usual an action of the Hecke algebra on $S_2(B)$, which preserves $\mathcal{M}$. As in the previous cases, one can define an η-invariant, η_g which satisfies

$$(\eta_g) \subseteq (\langle g, g \rangle)$$

and equality holds *provided* a suitable localization of $\mathcal{M}$ is free over the Hecke algebra. In this case, no transcendental periods appear, and the requisite freeness results are known in some cases but appear quite delicate in general.

Our point of view is going be to focus on the Petersson inner products, rather than on the canonical periods or the eta-invariants. We would like to understand how the Petersson inner products vary as we vary B over all the quaternion algebras that f transfers to. One advantage of the Petersson inner products seems to be that one can obtain relations between them without the delicate freeness results that seem to intervene when studying η-invariants. In the next section, we will describe what can be proved for quaternion algebras over $\mathbf{Q}$ in the weight 2 case.

3. Integral period relations for quaternion algebras over $\mathbf{Q}$

The main theorem below is not stated anywhere as such in the literature, but is a combination of various results. We state it in this way since this seems to be a rather elegant

[3] Forms of higher weight correspond to suitable vector-valued functions on $\mathrm{Cl}(M_B)$.

formulation that deals at once with definite and indefinite quaternion algebras. Further it should generalize to higher weights and also to Hilbert modular forms.

Theorem 3.1. *Let f be a modular form of weight 2 on $\Gamma_0(N)$ with N square-free. Let $\lambda \nmid 2N$ be a prime in K_f such that $\bar{\rho}_{f,\lambda}$ is irreducible. Let Σ_f be the set of primes dividing $N\infty$. Then there exists a function*

$$c : \Sigma_f \to \mathbf{C}, \qquad v \mapsto c_v,$$

with the property that if B is any quaternion algebra that f admits a Jacquet-Langlands transfer to, we have

$$\langle f_B, f_B \rangle \sim_\lambda \frac{L(1, \mathrm{ad}^0 f)}{\prod_{v \in \Sigma_B} c_v}$$

where f_B denotes a λ-adically normalized form on B corresponding to f and Σ_B is the set of places (including ∞ possibly !) where B is ramified. Further, if q is a finite prime in Σ_f, then $c_q \in \mathcal{O}_\lambda$ and $(c_q\mathcal{O}_\lambda)$ is the largest power of the maximal ideal in $\mathcal{O}_\lambda$ such that $\rho_{f,\lambda}$ mod $(c_q\mathcal{O}_\lambda)$ is unramified at q.

We will sketch a proof assuming for simplicity that the Fourier coefficients of f live in $\mathbf{Q}$ i.e. f corresponds to an isogeny class of elliptic curves over $\mathbf{Q}$. [4] Thus $\lambda = \ell$ is a rational prime. In this case, we define c_q for finite q to be the order of the component group of the Neron model of E at q of any curve E in this isogeny class. Since ℓ is not Eisenstein for f, this order is well defined up to ℓ-units. Also, from the Tate parametrization of E at q, we see that $(c_q\mathbf{Z}_\ell)$ satisfies the last condition in the statement of the theorem. At infinity, we define

$$c_\infty := \int_{E(\mathbf{C})} \omega_E \wedge \bar{\omega}_E$$

where ω_E is a Neron differential on E. Again, this is well defined up to ℓ-adic units if ℓ is not Eisenstein for f.

We first deal with the indefinite case, which involves two main ingredients:

(i) A result due to Ribet and Takahashi relating degrees of modular parametrizations by modular curves and Shimura curves.

(ii) A study of the Manin constant for the same parametrizations.

Here is an outline of (i) following Ribet-Takahashi[29]. Let V be an abelian variety over $\mathbf{Q}$ and q a prime at which V has semi-stable reduction i.e. the reduction mod q of the Neron model of V is an extension of an abelian variety by a torus. Let $T(V,q)$ denote this torus and let $\mathcal{X}(V,q)$ denote the character group $\mathrm{Hom}_{\bar{\mathbf{F}}_q}(T(V,q), \mathbf{G}_m)$. There is a canonical bilinear (monodromy) pairing

$$u_V : \mathcal{X}(V,q) \times \mathcal{X}(V^\vee, q) \to \mathbf{Z}.$$

[4] The same proof generalizes with a little more effort to the case where the Fourier coefficients generate a larger number field.

If V is the Jacobian of a curve (as we assume henceforth), then $V^\vee$ is canonically isomorphic to V and we get a bilinear pairing

$$u_V : \mathcal{X}(V,q) \times \mathcal{X}(V,q) \to \mathbf{Z}.$$

This pairing is not perfect in general; on the contrary, it induces an exact sequence of the form

$$0 \to \mathcal{X}(V,q) \to \mathrm{Hom}(\mathcal{X}(V,q),\mathbf{Z}) \to \Phi(V,q) \to 0$$

where $\Phi(V,q)$ denotes the group of components of the Neron model of V at q.

Let $J_0^{N_1}(N_2)$ denote the Jacobian of the Shimura curve corresponding to an Eichler order of level N_2 in a quaternion algebra of discriminant N_1. Suppose $N = DpqM$ for primes p and q. Let $J := J_0^D(pqM)$ and $J' := J_0^{Dpq}(M)$. Then there exist unique elliptic curves E and E' in the isogeny class corresponding to f that are *strong* Weil curves for J and J' respectively. Namely, there are maps

$$\xi : J \to E, \qquad \xi' : J' \to E'$$

such that any map from J (resp. J') to an elliptic curve in this isogeny class factors through ξ (resp. ξ'.) The composite $\xi \circ \xi^\vee$ (resp. $\xi' \circ \xi'^\vee$) is just multiplication on E (resp. E') by an integer δ (resp. δ'), called the modular degree.

We let

$$\xi_{*,\Phi} : \Phi(J,q) \to \Phi(E,q), \qquad \xi'_{*,\Phi} : \Phi(J',p) \to \Phi(E',p),$$

denote the induced maps on component groups. (Notice that we've used two different primes for J and J' !) Also let $J'' := J_0^D(qM)$. The key ingredient is the following fact, first proved by Ribet [27] when $D = 1$ and generalized by Buzzard [3] for $D > 1$.

Proposition 3.2. *There is an exact sequence*

$$0 \to \mathcal{X}(J',p) \xrightarrow{i} \mathcal{X}(J,q) \to \mathcal{X}(J'',q) \times \mathcal{X}(J'',q) \to 0,$$

which commutes with the action of the Hecke operators T_n for n prime to N. Further, the map i is compatible with the monodromy pairings on $\mathcal{X}(J',p)$ and $\mathcal{X}(J,q)$.

Let $\mathcal{L}$ be defined by

$$\mathcal{L} := \{x \in \mathcal{X}(J,q) : T_n x = a_n(f)x \text{ for all } n \nmid N\}.$$

i.e. the f-part of $\mathcal{X}(J,q)$. Likewise, define $\mathcal{L}' \subset \mathcal{X}(J,p)$, so that $\mathcal{L}' = \mathcal{L} \cap \mathcal{X}(J',p)$. Note that both $\mathcal{L}$ and $\mathcal{L}'$ are free $\mathbf{Z}$-modules of rank one. Hence the image of $\mathcal{L}$ in $\mathcal{X}(J'',q) \times \mathcal{X}(J'',q)$ is torsion, and consequently zero. Thus i induces an isomorphism $\mathcal{L}' \simeq \mathcal{L}$. Let us fix generators g, g' of $\mathcal{L}, \mathcal{L}'$ respectively such that $i(g') = g$.

Let x be a generator of $\mathcal{X}(E,q)$, and let

$$\xi^* : \mathcal{X}(E,q) \to \mathcal{X}(J,q), \qquad \xi_* : \mathcal{X}(J,q) \to \mathcal{X}(E,q),$$

denote the maps induced by $\xi, \xi^\vee$ respectively. The composite $\xi_* \circ \xi^*$ is just multiplication by the modular degree δ. Since $u_E(x,x) = c_q(E)$, we get

$$\delta c_q = u_E(x, \xi_*\xi^* x) = u_J(\xi^* x, \xi^* x) = n^2 u_J(g,g) \tag{4}$$

where $\xi^* x = ng$. Likewise
$$\delta' c'_p = m^2 u_{J'}(g', g'),$$
where $\xi'^* x' = mg'$ with x' a generator of $\mathcal{X}(E', q)$. Since $u_{J'}(g', g') = u_J(g, g)$, taking ratios gives:
$$\frac{\delta c_q}{n^2} = \frac{\delta' c'_p}{m^2}.$$
Now look at the commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}(J,q) & \longrightarrow & \mathrm{Hom}(\mathcal{X}(J,q),\mathbf{Z}) & \longrightarrow & \Phi(J,q) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi_*} & & \downarrow{\scriptstyle \mathrm{Hom}(\xi^*,\mathbf{Z})} & & \downarrow{\scriptstyle \xi_{*,\Phi}} & & \\ 0 & \longrightarrow & \mathcal{X}(E,q) & \longrightarrow & \mathrm{Hom}(\mathcal{X}(E,q),\mathbf{Z}) & \longrightarrow & \Phi(E,q) & \longrightarrow & 0. \end{array}$$
The left vertical map is surjective since $J \to E$ being optimal implies $\xi^\vee : E \to J$ is injective. Hence the cokernels of the two vertical maps on the right are identified: this implies $n = |\operatorname{coker} \xi_{*,\Phi}|$. Likewise, $m = |\operatorname{coker} \xi'_{*,\Phi}|$. Now one uses the following key fact also due to Ribet [25]: the action of the Hecke algebra on $\Phi(J, q)$ is Eisenstein. Hence $n \sim c_q$ and
$$\delta \sim \delta' \cdot \frac{c_q c_p}{m^2}.$$
(Here we have used again that ℓ is not Eisenstein for f, hence $c_p \sim c'_p$ and $c_q \sim c'_q$.) Now we know that m divides c_q; however by interchanging p and q, we find m divides c_p as well. Finally, one uses a trick to show that m divides c_r for all r, hence is not divisible by ℓ; if not, by Ribet's level-lowering theorem, f would be congruent mod ℓ to a form of weight 2 and level one, and such a form does not exist. The conclusion then is that
$$\delta \sim \delta' \cdot c_p c_q$$
and by induction
$$\delta(N^-, N^+) = \frac{\delta(1, N)}{\prod_{q|N^-} c_q}.$$
It remains to relate the modular degrees to the Petersson inner products. This requires a study of the Manin constant. Let us denote also by ξ the map $X_0(N) \to E$ obtained by composing with an embedding $X_0(N) \hookrightarrow J_0(N)$. Then for ω a Neron differental on E, the pull back $\xi^*(\omega)$ equals $2\pi i \omega_f$ on $X_0(N)$, up to a power of 2, where ω_f denotes an integrally normalized form. (This is due to Mazur and Raynaud in the semistable case: for a proof, see Abbes-Ullmo[1] and Ullmo[35].) A similar result holds for Shimura curve parametrizations, as is explained in §2.2.1 of [22]. Using again the fact that any two elliptic curves in the isogeny class are isogenous by an isogeny of degree prime to ℓ, one finds that
$$\langle f_B, f_B \rangle \sim \langle f, f \rangle \cdot \frac{\delta'}{\delta} \sim \frac{L(1, \mathrm{ad}^0 f)}{\prod_{q|N^-} c_q}.$$

Next we sketch the definite case.The key ingredient is the following proposition: (I learnt this from Pollack-Weston[19], who refer to Kohel[17], who in turn cites Takahashi[33] for this formulation, and explains that the result is a consequence of Thm. 4.7 and Thm. 4.10 of Buzzard[3], that generalize the corresponding theorems in Deligne-Rapoport.)

Proposition 3.3. *Let B be the definite quaternion algebra of discriminant N^-. For $q \mid N^-$, there is a canonical Hecke equivariant isomorphism*

$$\mathcal{M}_B \simeq \mathcal{X}(J,q),$$

where J is the Jacobian of the Shimura curve of level N^+q in the indefinite quaternion algebra of discriminant N^-/q. Further this isomorphism takes the inner product on $\mathcal{M}_B$ to the monodromy pairing on $\mathcal{X}(J,q)$.

It follows from the proposition and the proof of the indefinite case that

$$\langle f_B, f_B\rangle = u_J(g,g) = \frac{\delta c_q}{n^2} \sim \frac{\delta}{c_q} \sim \frac{\delta(1,N)}{\prod_{r|N^-} c_r} \sim \frac{L(1,\mathrm{ad}^0 f)}{\prod_{v|N^-\infty} c_v}.$$

Remark 3.4. If one is only interested in period relations for definite quaternion algebras, then all one needs is Prop. 3.3, equation (4) and the fact that $n \sim c_q$. Indeed, suppose B is a definite quaternion algebra with discriminant N^-, $N = N^+N^-$ and $q \mid N^-, r \mid N^+$. Let B' be the definite quaternion algebra with discriminant N^-r/q. Then

$$\langle f_B, f_B\rangle \sim \frac{\delta(N^-/q, N^+q)}{c_q}, \qquad \text{and} \qquad \langle f_{B'}, f_{B'}\rangle \sim \frac{\delta(N^-/q, N^+q)}{c_r},$$

hence

$$c_r \cdot \langle f_B, f_B\rangle \sim c_q \cdot \langle f_{B'}, f_{B'}\rangle.$$

Remark 3.5. For $\lambda \mid N$, we can still normalize forms canonically. On definite quaternion algebras, this is no problem at all, while on indefinite ones we use a minimal regular model of the Shimura curve. With this normalization, the result above continues to hold, except if $\lambda \mid q$, then $(c_q\mathcal{O}_\lambda)$ is interpreted as the largest power (λ^{n_q}) such that $A_f[\lambda^{n_q}]$ extends to a finite flat group scheme over $\mathcal{O}_\lambda$. For $\lambda|2$, again there is no problem in the definite case, while in the indefinite case the only ambiguity arises from the Manin constant.

Question 3.6. *This leaves open the question of what happens in the Eisenstein case. See Problem 1.*

4. The theta correspondence

In this section, we change focus somewhat and describe the basic constructions in the theta correspondence. Later we will show how a study of the arithmetic of theta liftings is related to the themes of the previous sections. We will be brief since this material is amply explained elsewhere. In particular, the expository article of Dipendra Prasad [21] is an excellent reference, and much of this section is a summary of the corresponding parts of that article.

Let k be a local field of characteristic not equal to 2. Let W be a finite dimensional symplectic space over k i.e. W is equipped with a non-degenerate alternating form $\langle\cdot,\cdot\rangle$. The Heisenberg group $H(W)$ is a nontrivial central extension of W by k,

$$0 \to k \to H(W) \to W \to 0,$$

defined as follows. We let $H(W)$ be the set of pairs (w,t) with $w \in W$ and $t \in K$, multiplication being defined by:

$$(w,t)\cdot(w',t') = (w+w', t+t' + \frac{1}{2}\langle w, w'\rangle).$$

The key fact underlying the theory is the following result of Stone and von Neumann: given any additive character

$$\psi : k \to \mathbf{C}^\times,$$

the group $H(W)$ admits a unique irreducible smooth representation $(\rho_\psi, \mathcal{S})$ on which k acts via ψ. In fact, given any decomposition

$$W = W_1 \oplus W_2$$

into a sum of maximal isotropic spaces, the representation ρ_ψ can be realized on the Schwartz space $\mathcal{S}(W_1)$. (This is the space of locally constant functions with compact support in the non-archimedean case, and the space of C^∞-functions of rapid decay in the archimedean case.) The formulas defining ρ_ψ on $\mathcal{S}(W_1)$ are:

$$\begin{aligned} \rho_\psi(w_1)f(x) &= f(x+w_1), \\ \rho_\psi(w_2)f(x) &= \psi(\langle x, w_2\rangle)f(x), \\ \rho_\psi(t)f(x) &= \psi(t)f(x), \end{aligned}$$

for $w_1, x \in W_1$, $w_2 \in W_2$ and $t \in k$.

Let $\mathbf{Sp}(W)$ denote the symplectic group of W i.e. the group of linear automorphisms of W preserving the form $\langle\cdot,\cdot\rangle$. Then $\mathbf{Sp}(W)$ operates on $H(W)$ by $g\cdot(w,t) = (gw,t)$. Thus for $g \in \mathbf{Sp}(W)$,

$$(w,t) \mapsto \rho_\psi(g\cdot(w,t))$$

is another representation of $H(W)$ on $\mathcal{S}$ which k acts via ψ. By the irreducibility of $(\rho_\psi, \mathcal{S})$, there exists an operator $\omega_\psi(g)$ on $\mathcal{S}$ unique up to scaling such that

$$\rho_\psi(gw,t)\cdot\omega_\psi(g) = \omega_\psi(g)\cdot\rho_\psi(w,t) \quad \text{for all} \quad (w,t) \in H(W).$$

Let $\mathbf{Mp}_\psi(W)$ denote the group of pairs (g, ω_ψ) satisfying the relation above, with multiplication just defined coordinatewise. It fits into an exact sequence

$$0 \to \mathbf{C}^* \to \mathbf{Mp}_\psi(W) \to \mathbf{Sp}(W) \to 0.$$

It comes equipped with a natural representation ω_ψ on $\mathcal{S}$ given by projection on the second factor, called the *Weil representation.*

We now recall the key notion of a *dual reductive pair*, due to Howe. This is a pair (G, G') of reductive subgroups of $\mathbf{Sp}(W)$ that are centralizers of each other. Let $\tilde{G}$ and $\tilde{G}'$ denote the preimages of G and G' in $\mathbf{Mp}(W)$. It can be shown that $\tilde{G}$ and $\tilde{G}'$ commute with each other. Hence the Weil representation can be used to construct a correspondence between representations of $\tilde{G}$ and $\tilde{G}'$ that occur in the restriction of ω_ψ to $\tilde{G}$ and $\tilde{G}'$ respectively. Here are some important examples of dual reductive pairs.

Example 4.1. Let W be a symplectic space and V an orthogonal space. Set $\mathbb{W} := W \otimes V$. Then $(\mathbf{Sp}(W), \mathbf{O}(V))$ is a dual reductive pair in $\mathbf{Sp}(\mathbb{W})$.

Example 4.2. Let K/k be a (possibly split) quadratic extension (with $a \mapsto \bar{a}$ the nontrivial involution of K/k), V (resp. W) be a right K-vector space equipped with a nondegenerate Hermitian (resp. skew-Hermitian) form. i.e.

$$\langle v_1\alpha_1, v_2\alpha_2\rangle = \bar{\alpha}_1\langle v_1, v_2\rangle\alpha_2, \qquad \langle v_2, v_1\rangle = \overline{\langle v_2, v_1\rangle},$$

$$\langle w_1\alpha_1, w_2\alpha_2\rangle = \bar{\alpha}_1\langle w_1, w_2\rangle\alpha_2 \qquad \langle w_2, w_1\rangle = -\overline{\langle w_2, w_1\rangle},$$

for $v_i \in V$, $w_i \in W$, $\alpha_i \in K$. Let $\mathbb{W} := V \otimes W$ with alternating form given by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = \mathrm{tr}_{K/k}(\langle v_1, v_2\rangle \cdot \langle w_1, w_2\rangle).$$

Then $(\mathbf{U}(V), \mathbf{U}(W))$ is a dual reductive pair in $\mathbf{Sp}(\mathbb{W})$.

We now describe the corresponding global objects. Let k be a number field, $\mathbb{A}_k$ the adele ring of k and W a symplectic vector space over k. Let $H(W)$ denote the Heisenberg group attached to W, viewed as an algebraic group over k and $\psi : \mathbb{A}_k/k \to \mathbf{C}^*$ a nontrivial additive character. Also let $W = W_1 \oplus W_2$ be a complete polarization. In exactly the same manner as the local case, one can define a global metaplectic group $\mathbf{Mp}(W)(\mathbb{A}_k)$[5] which sits in an exact sequence

$$0 \to \mathbf{C}^* \to \mathbf{Mp}(W)(\mathbb{A}_k) \to \mathbf{Sp}(W)(\mathbb{A}_k) \to 0,$$

and admits a (global) Weil representation on $\mathcal{S}(W_1(\mathbb{A}_k))$. A key theorem due to Weil is that the covering $\mathbf{Mp}(W)(\mathbb{A}_k) \to \mathbf{Sp}(W)(\mathbb{A}_k)$ splits canonically over $\mathbf{Sp}(W)(k)$. Indeed, Weil defines the theta distribution on $\mathcal{S}(W_1(\mathbb{A}))$:

$$\Theta(\varphi) = \sum_{x \in W_1(k)} \varphi(x),$$

and shows that Θ is (up to scaling) the unique linear form on $\mathcal{S}(W_1(\mathbb{A}))$ that is $H(W)(k)$-invariant. Now it is easily checked that for any $g \in \mathbf{Mp}(W)(k)$ i.e. the preimage of $\mathbf{Sp}(W)(k)$ in $\mathbf{Mp}(W)(\mathbb{A}_k)$, the linear form

$$\varphi \mapsto \Theta(\omega_\psi(g)\varphi)$$

is also $H(W)(k)$-invariant, hence there is a scalar $\lambda_g \in \mathbf{C}^*$ such that

$$\Theta(\omega_\psi(g)\varphi) = \lambda_g\Theta(\varphi).$$

The assignment $g \mapsto \lambda_g$ is then a homomorphism $\mathbf{Mp}(W)(k) \to \mathbf{C}^*$ which splits the exact sequence

$$0 \to \mathbf{C}^* \to \mathbf{Mp}(W)(k) \to \mathbf{Sp}(W)(k) \to 0.$$

[5]This notation is somewhat misleading because $\mathbf{Mp}(W)(\mathbb{A})$ is not the adelic points of an algebraic group.

Finally, we will recall the definition of global theta lifts. Pick $\varphi \in \mathcal{S}(W_1(\mathbb{A}))$ and define a function $\theta_\varphi(g)$ for $g \in \mathbf{Mp}(W)(\mathbb{A})$ by

$$\theta_\varphi(g) = \Theta(\omega_\psi(g)\varphi).$$

This defines an automorphic function on $\mathbf{Mp}(W)(\mathbb{A})/\mathbf{Mp}(W)(k)$. Let (G, G') be a dual reductive pair in $\mathbf{Sp}(W)$ (i.e. G and G' are reductive algebraic groups over k that are centralizers of each other in $\mathbf{Sp}(W)$). Also for any algebraic subgroup H of $\mathbf{Sp}(W)$, let $\tilde{H}(\mathbb{A})$ and $\tilde{H}(k)$ denote the inverse images of $H(\mathbb{A})$ and $H(k)$ in $\mathbf{Mp}(W)(\mathbb{A})$. Since $\tilde{G}(\mathbb{A})$ and $\tilde{G}'(\mathbb{A})$ commute with each other, we can restrict the function θ_φ to a function $\theta_\varphi(g, g')$ on $\tilde{G}(\mathbb{A}) \times \tilde{G}'(\mathbb{A})$, called the theta kernel. This provides an integrating kernel to transfer automorphic forms from $\tilde{G}$ to $\tilde{G}'$ and in the opposite direction as well. For f (resp. f') an cuspform on $G(k)\backslash\tilde{G}(\mathbb{A})$ (resp. $G'(k)\backslash\tilde{G}'(\mathbb{A})$), we define

$$\theta_\varphi(f)(g') = \int_{G(k)\backslash\tilde{G}(\mathbb{A})} f(g)\theta_\varphi(g, g')dg,$$

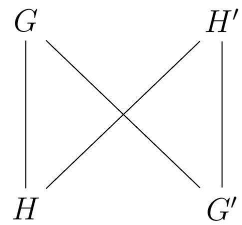
$$\theta_\varphi(f')(g) = \int_{G'(k)\backslash\tilde{G}(\mathbb{A})} f'(g)\theta_\varphi(g, g')dg',$$

where dg and dg' are suitably chosen measures. The span of $\theta_\varphi(f)$ as f varies over all the forms in a cuspidal representation π and φ varies over all the elements of $\mathcal{S}(V_1(\mathbb{A}))$ will be denoted $\Theta(\pi, \psi)$.

Remark 4.3. It is a very useful fact that for a dual reductive pair (G, G'), the metaplectic cover splits over $G(\mathbb{A})$ unless (G, G') is the pair $(\mathbf{Sp}(W), \mathbf{O}(V))$ in $\mathbf{Sp}(W \otimes V)$ and V is odd-dimensional. This will be the case in all the examples we consider below. Consequently, we can replace $\tilde{G}$ and $\tilde{G}'$ by G and G' in the above discussion. There is however a tricky issue having to do with picking the choice of splitting, since it may not be unique. For example, it is not clear that one can pick a splitting over $G(\mathbb{A})$ such that the image of $G(\mathbb{A})$ contains the image of $G(k)$ under the previously chosen splitting of $\mathbf{Sp}(W)(k)$. We will ignore this issue here.

Remark 4.4. An important question in the theory of the theta correspondence is to determine when $\Theta(\pi, \psi)$ is nonzero. This can be quite subtle: for example, it could involve both local conditions (epsilon factors) and global conditions (nonvanishing of L-values). It is known that if $\Theta(\pi, \psi)$ is nonzero and consists of cusp forms, then it is irreducible.

Finally, we discuss the notion of *seesaw pairs* due to Kudla [18]. Two dual reductive pairs (G, G') and (H, H') in $\mathbf{Sp}(W)$ are said to form a seesaw pair if $H \subset G$ and $G' \subset H'$. We will denote this



The usefulness of this definition stems from the following result, whose proof follows from unwinding the definition an a change of variables.

Proposition 4.5. *(Seesaw duality) Suppose f is a cusp form on H and f' a cusp form on G'. Then*

$$\langle \theta_\varphi(f), g\rangle_{G'} = \langle f, \overline{\theta_\varphi(f')}\rangle_H$$

where we view $\theta_\varphi(f)$ as a function on $\tilde{H}'(\mathbb{A})$ restricted to $\tilde{G}'(\mathbb{A})$ and likewise $\theta_\varphi(f')$ as a function on $\tilde{G}(\mathbb{A})$ restricted to $\tilde{H}(\mathbb{A})$.

5. Arithmetic of the Shimizu lift and Waldspurger's formula

We now return to the problem of understanding Petersson inner products using the theta correspondence. The key input will be the the Shimizu lift which gives an explicit realization of the Jacquet-Langlands correspondence.

Let B be a quaternion algebra over $\mathbf{Q}$. We let $V = B$ viewed as a quadratic space with inner product

$$\langle x, y\rangle = xy^i + yx^i$$

where i denotes the main involution on B. Let W denote a 2 dimensional symplectic space over $\mathbf{Q}$ and consider the dual pair $(\mathbf{Sp}(W), \mathbf{O}(V))$. In fact, it is more convenient to work with the connected components of the similitude groups $\mathbf{GSp}(W) = \mathbf{GL}_2$ and $\mathbf{GO}(V)$. There is a map

$$B^\times \times B^\times \to \mathbf{GO}(V)$$
$$(\alpha, \beta) \rightsquigarrow (x \mapsto \alpha x \beta^{-1}),$$

which surjects onto the connected component of $\mathbf{GO}(V)$, and whose kernel is $\mathbf{Q}^\times$ embedded diagonally. Thus an automorphic representation of $\mathbf{GO}(V)^0$ is of the form $\pi_1 \otimes \pi_2$ where the π_i are automorphic representations of $B_A^\times$ such that the product of their central characters is trivial.

The theory of theta lifting from the previous section can be extended to similitude groups, see [9] for example. The key result is then the following, due to Shimizu:

Theorem 5.1. *Suppose π is an automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ and ψ any additive character of $\mathbf{Q}\backslash\mathbb{A}$. If π does not transfer to $B^\times(\mathbb{A})$, then $\Theta(\pi, \psi) = 0$. If π does transfer to an automorphic representation π_B on $B^\times(\mathbb{A})$, then*

$$\Theta(\pi, \psi) = \pi_B \otimes \pi_B^\vee.$$

Suppose that f is a newform in $S_k(\Gamma_0(N))$ and let π denote the corresponding automorphic representation of $\mathbf{GL}_2(\mathbb{A})$, so that $\pi^\vee \simeq \pi$. We may view f in the usual way as an element of π, and likewise f_B as an element of π_B. (See appendix B for more details on this.) To compare periods, one needs to pick an explicit Schwartz function $\varphi \in \mathcal{S}(V_1(\mathbb{A}))$ such that

$$\theta_\varphi(f) = \alpha \cdot (f_B \times f_B),$$

for a nonzero scalar α. In fact, it is easier to study the theta lift in the opposite direction, since then one can compute the Fourier coefficients of the lifted form. One finds with a good choice of φ (see [38]) that

$$\theta_\varphi(f_B \times f_B) = \langle f_B, f_B \rangle \cdot f.$$

Using seesaw duality, it follows that

$$\alpha = \frac{\langle f, f\rangle}{\langle f_B, f_B\rangle}.$$

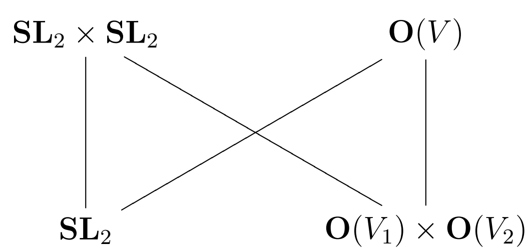
To study the arithmetic of the scalar α, we follow Harris-Kudla [10] in using Waldspurger's work [40] relating period integrals to L-values. Namely, we pick a torus corresponding to an imaginary quadratic field K in $B^\times$ and compute the periods of $\theta_\varphi(f)$ along $K^\times \times K^\times$, twisted by a Hecke character χ of K of infinity type $(k, 0)$. If we write

$$B = K \oplus K\mathbf{j} = V_1 \oplus V_2$$

as the sum of two orthogonal spaces, then the map $B^\times \times B^\times \to \mathbf{GO}(V)^0$ sends $K^\times \times K^\times$ to $\mathbf{G}(\mathbf{O}(V_1) \times \mathbf{O}(V_2))^0 = \mathbf{G}(K^\times \times K^\times)$. This map is easily seen to be

$$(x, y) \mapsto (xy^{-1}, x\bar{y}^{-1}).$$

Also there is a seesaw pair



Since π has trivial central character, the integral

$$L_\chi(f_B) := \int_{\mathbb{A}_K^\times} f_B \cdot \chi$$

is trivial unless the central character $\xi_\chi := \chi|_{\mathbb{A}_{\mathbf{Q}}^\times}$ is trivial. So we may assume this is the case, and it follows then that

$$\chi(x)\chi(y) = \chi(x)\chi(\bar{y})^{-1} = \chi(x\bar{y}^{-1}).$$

Another application of seesaw duality gives

$$\alpha \cdot L_\chi(f_B)^2 = \int_{\mathbf{GL}_2(\mathbb{A})} f(g)\theta_\varphi((\mathbf{1},\chi))(g)dg.$$

If the Schwartz function φ factors as

$$\varphi = \varphi_1 \otimes \varphi_2 \in \mathcal{S}(V_1(\mathbb{A})) \otimes \mathcal{S}(V_2(\mathbb{A})),$$

then the integral above is

$$\int_{\mathbf{GL}_2(\mathbb{A})} f(g)\theta_{\varphi_1}(\mathbf{1})\theta_{\varphi_2}(\chi)(g)dg. \tag{5}$$

For simplicity, we will assume this is the case, though in general, φ is only a sum of pure tensors and so one gets a sum of such integrals.

To analyze the arithmetic properties of α, we first analyze the integral $L_\chi(f_B)$. One observes that $L_\chi(f_B)$ is just a sum of values of f_B at conjugates of a CM point on X_B, twisted by the values of χ (so that such a sum makes sense !) The curve X_B is a coarse moduli space for abelian surfaces with endomorphisms by an order in B. As in the case of usual modular forms, one can think of modular forms on X_B as functions of pairs (A,ω), where A is such an abelian surface and ω is a differential form on A (in fact an element of $\wedge^2\Omega^1_A$) and satisfying an appropriate transformation law. Now CM points on X_B correspond to products $E \times E$ where E is an elliptic curve with CM by an imaginary quadratic field K. Since CM elliptic curves have potentially good reduction and f_B is algebraic and even integrally normalized, its values at CM points suitably normalized should be algebraic integers. Further, for any prime p, there exist a choice of CM point such that the value of f_B is a p-unit. It follows from this discussion that the arithmetic properties of α are controlled entirely by those of the triple integral (5).

In the present case, it turns out that f and $\theta_1 := \theta_{\varphi_1}(\mathbf{1})$ are holomorphic of weights k and 1 respectively, while $\theta_{\varphi_2}(\chi)$ is the complex conjugate of a holomorphic form θ_χ of weight $k+1$. So the triple integral is of the form

$$\langle f\theta_1, \theta_\chi\rangle. \tag{6}$$

This can be analyzed by decomposing $f\theta_1$ as a sum of eigenforms. Suppose

$$f\theta_1 = g + g'$$

where g is the projection of $f\theta_1$ onto the old space spanned by θ_χ. Then

$$\langle f\theta_1, \theta_\chi\rangle = \langle g, \theta_\chi\rangle.$$

On the other hand, the Petersson inner product $\langle \theta_\chi, \theta_\chi\rangle$ can be related to the L-value $L(\chi(\chi^\rho)^{-1}, 1)$ which in turn is known by work of Shimura to be algebraic up to the correct CM period. This turn out to be enough to show algebraicity of α and with some additional care, even the rationality of α over K_f (see [10] §15). The integrality is somewhat more subtle, and requires a precise control of the denominators of g. This can be accomplished using the Iwasawa main conjecture for K, a theorem of Rubin [30]; the reader is referred to [22] for the details of this argument.

The upshot of the arguments sketched above is that $\alpha := \langle f, f\rangle/\langle f_B, f_B\rangle$ is p-integral (under some local conditions on p.) Note that one advantage of the current argument is that works equally well when the weight is not 2; the analog of the geometric arguments (especially the Tate conjecture) of Sec. 3 are not known in this setting.

The connection with level-lowering congruences is seen most easily through the formula of Waldspurger mentioned above. Indeed, the form θ_1 is identified with an Eisenstein series by the Siegel-Weil formula and the integral (6) is then identified with the value at the center of the integral representation of the Rankin-Selberg L-function $L(s, f \times \theta_\chi)$. If f satisfies a congruence mod p with a form g of level N^-/q, then g satisfies a level-raising congruence at q mod p and this imposes a condition on the Hecke eigenvalue of g at q. One can check that this condition forces the L-value above to be divisible by p. (See [22] §5.2 for more details.)

Remark 5.2. One would like something more precise, namely a factorization of α as a product of Tamagawa numbers c_q. It seems plausible that this will follow by combining the methods described in this section with those of Sec. 3 and a systematic use of p-adic families.

6. Hilbert modular forms, Shimura's conjecture and a refined version

We now move to the setting of Hilbert modular forms. It will be more convenient to simply use the language of automorphic representations. So suppose that F is a totally real field, and π is an automorphic representation of $\mathbf{GL}_2(\mathbb{A}_F)$, that is holomorphic discrete series of parallel weight 2. More generally, we could assume that the weight of π is equal to $(k_1, \ldots, k_d)$ with

$$k_1 \equiv \cdots \equiv k_d \mod 2,$$

but for simplicity we will restrict ourselves to the parallel weight 2 case in these notes. Let us also assume for simplicity that the conductor of π is a square-free ideal $\mathfrak{n}$ in $\mathcal{O}_F$. If B is any quaternion algebra over F with discriminant dividing $\mathfrak{n}\infty_1 \cdots \infty_d$, then by the Jacquet-Langlands correspondence π transfers to an automorphic representation π_B on $B^\times(\mathbb{A}_F)$. We can also fix an arithmetic form f_B in π_B exactly as before. Namely, f_B corresponds to a section of an automorphic vector bundle (in fact line bundle in the parallel weight 2 case) on the Shimura variety X_B associated to B. This variety and the associated line bundle have canonical models over a reflex field which extend to smooth models outside the primes dividing $\mathfrak{n}$. So at least for $\lambda \nmid \mathfrak{n}$, one can define the notion of a λ-adically normalized form.

We now recall Shimura's conjecture for arithmetically normalized forms.

Conjecture 6.1. *There exists a function $c : \Sigma_\infty \to \mathbf{C}^\times$ such that*

$$\langle f_B, f_B\rangle \sim_{\overline{\mathbf{Q}}^*} \prod_{\substack{v\in\Sigma_\infty \\ v\notin\Sigma_B}} c_v,$$

for all quaternion algebras B that f transfers to.

Shimura proved several results in this direction, for example that if B and B' have complementary ramification at the infinite places, then

$$\langle f, f\rangle \sim_{\overline{\mathbf{Q}}^*} \langle f_B, f_B\rangle \cdot \langle f_{B'}, f_{B'}\rangle.$$

and as a consequence of this that $\langle f_B, f_B\rangle$ (up to $\overline{\mathbf{Q}}^*$) only depends on the set of archimedean places where B is ramified.

Shimura's conjecture was proven by Michael Harris [8] under the following hypothesis:

Hypothesis (*): There exist at least one finite place v at which π_v is discrete series.

In the next section, we will outline the main ideas in his proof. The reason the hypothesis (*) is necessary is because the proof uses an induction on the number of infinite places where B is ramified and it becomes necessary to know that given any subset Σ of the infinite places, that there is a quaternion algebra B' such that the set of infinite places where B' is ramified is exactly Σ. The condition (*) was relaxed somewhat by work of Yoshida [39] using base change. Namely, as explained at the end of [39], Thm. 6.8 of loc. cit. can be used to relax condition (*), provided the weights satisfy $k_\tau \geq 3$. In Sec. 8, we shall outline a new method to approach this problem that seems to circumvent the need for this hypothesis. [6]

For the moment, we'd like to formulate a more precise version of Shimura's conjecture that works up to λ-adic units. Note that it follows from the statement of Shimura's conjecture that

$$L(1, \mathrm{ad}^0\pi) \sim_{\overline{\mathbf{Q}}^*} \prod_{v\in\Sigma_\infty} c_v$$

and hence

$$\langle f_B, f_B\rangle \sim_{\overline{\mathbf{Q}}^*} \frac{L(1, \mathrm{ad}^0\pi)}{\prod_{\substack{v\in\Sigma_\infty \\ v\in\Sigma_B}} c_v}.$$

Comparing this with Thm. 3.1, we are lead naturally to the following conjecture.

Conjecture 6.2. *Let $\Sigma(\pi)$ denote the set of places v of F for which the local component π_v is discrete series. Then there exists a function $c : \Sigma(\pi) \to \mathbf{C}$ such that*

$$\langle f_B, f_B\rangle \sim_\lambda \frac{L(1, \mathrm{ad}^0\pi)}{\prod_{v\in\Sigma_B} c_v}.$$

Remark 6.3. (1) This conjecture first appeared in [23].

(2) For infinite places v, one should expect the c_v to be transcendental and also algebraically independent unless π is special, for eg. a base change from a smaller field.

(3) On the other hand, for finite places v, one should expect that c_v is a λ-adic integer that counts level-lowering congruences for π at v. i.e. $(c_v\mathcal{O}_\lambda)$ is the largest power of the maximal ideal of $\mathcal{O}_\lambda$ such that $\rho_{f,\lambda} \mod (c_v\mathcal{O}_\lambda)$ is unramified at v.

[6] In the course of writing these notes, I found a way to modify Harris' original proof so that it also applies without assuming hypothesis (*) ! I will describe this at AWS, and also eventually include it in the next section.

7. Unitary groups and Harris' proof of algebraicity

In this section, we shall outline the main steps in the Harris' proof of Shimura's conjecture. The main idea is to study the arithmetic of theta lifts between unitary groups. Let B be a quaternion algebra over F and $E \hookrightarrow B$ be an embedding of E in B. Write

$$B = E + E\mathbf{j}, \tag{7}$$

where this is an orthogonal decomposition for the norm form, so that $\mathrm{tr}(\mathbf{j}) = 0$ and $x\mathbf{j} = \mathbf{j}x^\rho$ for $x \in E$. Then B is a two-dimensional Hermitian space over E, with inner product

$$\langle x, y\rangle = \mathrm{pr}(x^\rho y),$$

where $\mathrm{pr} : B \to E$ denotes the projection onto the first factor in equation (7). Note that we can make B into a skew-Hermitian space if we wish by picking an element $i \in E^\times$ with $\mathrm{tr}(i) = 0$ and setting

$$\langle\langle x, y\rangle\rangle = i\langle x, y\rangle.$$

The associated unitary group for the form $\langle\langle \cdot, \cdot\rangle\rangle$ is the same as that for $\langle \cdot, \cdot\rangle$. Let us denote the unitary similitude group $\mathbf{GU}_E(B)$. Since $\mathrm{tr}_{E/F} \circ \langle \cdot, \cdot\rangle$ is just the usual trace form on B, we see that

$$\mathbf{GU}_E(B) \hookrightarrow \mathbf{GO}(B),$$

and is identified with the subgroup of E-linear elements in $\mathbf{GO}(B)$. Recall that $\mathbf{GO}(B)$ is the semidirect product of its connected component $\mathbf{GO}(B)^0$ with the group of order two generated by $x \mapsto x^*$, and that $\mathbf{GO}(B)^0$ is identified with $(B^\times \times B^\times)/F^\times$ via the map

$$\rho(\alpha, \beta)x = \alpha x \beta^{-1}.$$

Note that ρ identifies $(B^\times \times K^\times)/F^\times$ with $\mathbf{GU}_K(B)$. Thus an automorphic representation of $\mathbf{GU}_E(B)$ is identified with a pair consisting of an automorphic representation π of $B^\times$ and a Hecke character χ of $E^\times$ such that $\xi_\pi \cdot \xi_\chi = 1$.

This section needs to be completed.

8. Quaternionic unitary groups

In this section, we will present an outline of some ongoing joint work with Ichino [14] which gives a new approach to this question and that seems well suited to study integrality questions. The starting point for this was a different question, which I will now outline, that has to do with the relation between period integrals of automorphic forms and special values of L-functions.

Recall that there are two well-known formulae of this kind.

(1) First, let π be an automorphic representation on $\mathbf{GL}_2(\mathbb{A}_F)$, E a quadratic extension (say CM extension) and χ a Hecke character of $\mathbb{A}_E^\times$ satisfying

$$\xi_\pi \cdot \xi_\chi = 1,$$

where ξ_π and ξ_χ denote the central characters of π and χ respectively. i.e. $\xi_\chi = \chi|_{\mathbb{A}_F^\times}$. The L-function $L(s,\pi_E,\chi) = L(s,\pi\times\pi_\chi)$ is then self-dual. Suppose that the corresponding epsilon factor $\varepsilon(\frac{1}{2},\pi,\chi) = 1$. Then a beautiful theorem of Tunnell-Saito-Waldspurger says that there is a unique quaternion algebra B with $\Sigma_B \subset \Sigma(\pi)$ (so that π admits a Jacquet-Langlands transfer π_B to $B^\times$), a form $f_B \in \pi_B$ and an embedding $E \hookrightarrow B$ such that

$$\frac{\left|\int f_B|_{E^\times(\mathbb{A})}\cdot\chi\right|^2}{\langle f_B, f_B\rangle} = \frac{L(\frac{1}{2},\pi_E,\chi)}{L(1,\mathrm{ad}^0\pi)}.$$

We remind the reader that equality here is only up to nonzero explicit factors that are relatively unimportant. The quaternion algebra B is determined by the local epsilon factors, namely for all places v, we have

$$\varepsilon_v(B) = \xi_{\pi,v}(-1)\varepsilon_v(\frac{1}{2},\pi_E,\chi).$$

(2) Next, suppose π_1, π_2, π_3 are three automorphic representations of $\mathbf{GL}_2(\mathbb{A}_F)$ satisfying

$$\xi_{\pi_1}\cdot\xi_{\pi_2}\cdot\xi_{\pi_3} = 1.$$

Then the triple product L-function $L(s,\pi_1\times\pi_2\times\pi_3)$ is self-dual. Suppose that the epsilon factor $\varepsilon(\frac{1}{2},\pi_1\times\pi_2\times\pi_3)$ is 1. Then a formula of Harris-Kudla (building on work of Dipendra Prasad, and extended by Watson [38], Ichino-Ikeda) shows that there is a unique quaternion algebra B such that

- All the π transfer to $B^\times$
- There exist $f_i \in \pi_{i,B}$ such that

$$\frac{\left|\int_{B_{\mathbf{Q}}^\times\backslash B_{\mathbb{A}}^\times} f_{1,B}\cdot f_{2,B}\cdot f_{3,B}\right|^2}{\langle f_{1,B},f_{1,B}\rangle\langle f_{2,B},f_{2,B}\rangle\langle f_{3,B},f_{3,B}\rangle} = \frac{L(\frac{1}{2},\pi_1\times\pi_2\times\pi_3)}{L(1,\mathrm{ad}^0(\pi_1))L(1,\mathrm{ad}^0(\pi_2))L(1,\mathrm{ad}^0(\pi_3))}.$$

Again, the quaternion algebra B is determined by a local epsilon factor:

$$\varepsilon_v(B) = \varepsilon_v(\frac{1}{2},\pi_1\times\pi_2\times\pi_3).$$

One would like to understand how these formulae are related, in a situation in which the degree 8 triple product L-function splits as a product of two Rankin-Selberg L-functions. This happens for instance when

$$\pi_1 = \pi, \qquad \pi_2 = \pi_{\eta_1}, \qquad \pi_3 = \pi_{\eta_3},$$

where η_1 and η_2 are two Hecke characters of E such that

$$\xi_\pi\cdot\xi_{\eta_1}\cdot\xi_{\eta_2} = 1.$$

Let $\chi_1 := \eta_1\cdot\eta_2$ and $\chi_2 = \eta_1\cdot\eta_2^\rho$ where ρ is the nontrivial element in $\mathrm{Gal}(E/F)$. Then

$$L(s,\pi_1\times\pi_2\times\pi_3) = L(s,\pi_E,\chi_1)\cdot L(s,\pi_E,\chi_2),$$

so that the two formulas above when combined give a formula of the sort

$$\left|\int_{B_{\mathbf{Q}}^\times\backslash B_{\mathbb{A}}^\times} f_{1,B}\cdot f_{2,B}\cdot f_{3,B}\right|^2 = \left|\int f_{B_1}|_{E^\times(\mathbb{A})}\cdot\chi_1\right|^2\cdot\left|\int f_{B_2}|_{E^\times(\mathbb{A})}\cdot\chi_2\right|^2, \tag{8}$$

up to normalizing factors. Here B, B_1 and B_2 could be different quaternion algebras. One quickly checks using the epsilon factor conditions above that they must satisfy the relation

$$B = B_1\cdot B_2$$

in the Brauer group.

Question: Can one prove a relation of the sort in (8) without using the Rankin-Selberg and triple product formulae ?

The key construction that enables one to do this is the following. Recall that the E embeds in both B_1 and B_2. Let us fix these embeddings and write

$$B_1 = E + E\mathbf{j}_1 \qquad B_2 = E + E\mathbf{j}_2$$

where these are orthogonal decompositions for the norm form. Thus $x\mathbf{j}_i = \mathbf{j}_i x^\rho$ for $i = 1, 2$ and $x \in E$ and $\mathrm{tr}(\mathbf{j}_1) = \mathrm{tr}(\mathbf{j}_2) = 0$. Define elements $J_1, J_2 \in E$ by $J_1 := \mathbf{j}_1^2, J_2 := \mathbf{j}_2^2$ and let $\mathrm{pr}_1 : B_1 \to E$ and $\mathrm{pr}_2 : B_2 \to E$ denote the projections onto the first factor. As in the previous section, we can view B_1 and B_2 as right Hermitian K-spaces, with inner product

$$(x, y) = \mathrm{pr}_j(x^*y), \qquad \text{for } x, y \in B_j.$$

Now set $V := B_1 \otimes_E B_2$ so that V is naturally a right Hermitian E-space. In fact, we shall modify this inner product to make it skew-Hermitian. Namely, pick an element $i \in E$ with $\mathrm{Tr}(i) = 0$ and define a skew-Hermitian pairing $(\cdot,\cdot)$ on V by

$$(x_1 \otimes x_2, y_1 \otimes y_2) = i(x_1, y_1)_1(x_2, y_2)_2.$$

Let $J := J_1J_2$ and let B be the quaternion algebra defined by the relations

$$B = E + E\mathbf{j}, \qquad \mathbf{j}^2 = J, \qquad x\mathbf{j} = \mathbf{j}x^\rho \text{ for } x \in E.$$

Then $B = B_1 \cdot B_2$ in the Brauer group. The right E-module V can in fact be given the structure of a right B-module, by setting

$$(1\otimes 1)\cdot\mathbf{j} = (\mathbf{j}_1\otimes\mathbf{j}_2), \qquad (\mathbf{j}_1\otimes 1)\cdot\mathbf{j} = J_1(1\otimes\mathbf{j}_2),$$

$$(1\otimes\mathbf{j}_2)\cdot\mathbf{j} = J_2(\mathbf{j}_1\otimes\mathbf{j}_2), \qquad (\mathbf{j}_1\otimes\mathbf{j}_2)\cdot\mathbf{j} = J(1\otimes 1).$$

It is tedious but straightforward to check that this gives a right action of B on V. Finally, one can define a B-skew Hermitian form $\langle,\rangle$ on V with the property that

$$\mathrm{pr}\circ\langle\cdot,\cdot,\rangle = (\cdot,\cdot),$$

where pr : $B \to E$ denotes the projection onto the first factor. In this way, V has the structure of a 2-dimensional right skew-Hermitian B-module. We will use the unitary group $\mathbf{GU}_B(V)$ or rather its connected component

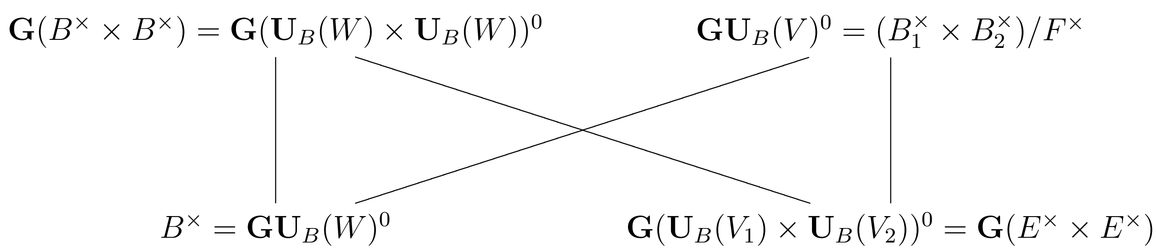
$$\mathbf{GU}_B(V)^0 = (B_1^\times \times B_2^\times)/F^\times,$$

where $F^\times$ embeds in $B_1^\times B_2$ by $\lambda \mapsto (\lambda^{-1}, \lambda)$ and an element (α_1, α_2) of $B_1^\times \times B_2^\times$ acts on V by left-multiplication.

Let $W = B$ thought of as a left Hermitian B-space with inner product $\langle x, y\rangle = x^* y$. We consider the theta correspondence from $B^\times = \mathbf{GU}_B(W)$ to $\mathbf{GU}_B(V)^0$. Note that an automorphic representation of $\mathbf{GU}_B(V)^0$ is identified with a pair (π_1, π_2), where π_j is an automorphic representation of B_j such that $\xi_{\pi_1} = \xi_{\pi_2}$. One can then show that

$$\theta(\pi_B) = \pi_{B_1} \times \pi_{B_2}, \tag{9}$$

as one would expect. Further, there is a see-saw pair of the form



The equality of the periods in (8) then follows from a simple application of Kudla's seesaw identity, once one understands the theta lifts in both direction.

The application to Petersson inner products comes form analyzing the arithmetic properties of the theta lift (9). The point is that for a nice canonical choice of theta function, we have

$$\theta(f_B) = \alpha(B_1, B_2) f_{B_1} \times f_{B_2},$$

for some scalar α_{B_1,B_2} ith f_B, f_{B_1} and f_{B_2} arithmetically normalized. For simplicity, let us assume that α_{B_1,B_2} is real. To relate the Petersson inner products, one uses the Rallis inner product formula which in this case takes the form

$$\alpha(B_1,B_2)^2 \langle f_{B_1}, f_{B_1}\rangle \langle f_{B_2}, f_{B_2}\rangle = \langle \theta(f_B), \theta(f_B)\rangle = \langle f_B, f_B\rangle \cdot L(1, \mathrm{ad}^0 \pi). \tag{10}$$

If conjecture A is true, then we get

$$\begin{aligned}\alpha(B_1,B_2)^2 &= \frac{\langle f_B, f_B\rangle \cdot L(1,\mathrm{ad}^0\pi)}{\langle f_{B_1}, f_{B_1}\rangle \cdot \langle f_{B_2}, f_{B_2}\rangle} \\ &= \frac{\prod_{v\in\Sigma_{B_1}} c_v \cdot \prod_{v\in\Sigma_{B_2}} c_v}{\prod_{v\in\Sigma_B} c_v} \\ &= \left(\prod_{v\in\Sigma_{B_1}\cap\Sigma_{B_2}} c_v\right)^2\end{aligned}$$

This leads to the following conjecture on arithmeticity of theta lifts.

Conjecture 8.1. *(Conjecture B)* [14]

(1) *If $\Sigma_{B_1}^\infty \cap \Sigma_{B_2}^\infty = \emptyset$, then $\alpha(B_1,B_2)$ is algebraic and is a λ-adic integer.*
(2) *If further $\Sigma_{B_1}\cap\Sigma_{B_2} = \emptyset$, then $\alpha(B_1,B_2)$ is a λ-adic unit.*

We have just seen that Conjecture A implies Conjecture B. We shall now show that the opposite implication is true as well, namely Conjecture B implies Conjecture A. To do so, we need to define suitable factors c_v. First, let $S\subset\Sigma(\pi)$ be any subset of $\Sigma(\pi)$ with $|S|$ even. For such S define

$$c_S := \frac{L(1,\mathrm{ad}^0\pi)}{\langle f_{B_S}, f_{B_S}\rangle}$$

where B_S is the unique quaternion algebra with discriminant S. Note that if S and T are two such sets of even cardinality that are disjoint, then

$$c_{S\sqcup T} = c_S \cdot c_T \tag{11}$$

as follows easily from equation (10) and Conjecture B. Now, we may assume that $\Sigma(\pi)$ has at least three elements. For any $v\in\Sigma(\pi)$, we pick any two other elements $s,t\in\Sigma(\pi)$ and define c_v by

$$c_v^2 = \frac{c_{vs}\cdot c_{vt}}{c_{st}}.$$

Let us show this is independent of the choice of s and t. It is enough to show that we may replace t by some $r\in\Sigma(\pi)$ which is distinct from v, s and t. Namely, we want to show

$$\frac{c_{vs}\cdot c_{vt}}{c_{st}} = \frac{c_{vs}\cdot c_{vr}}{c_{sr}} \quad \text{i.e. } c_{vt}\cdot c_{sr} = c_{vr}\cdot c_{st}.$$

But this follows since both the LHS and RHS of the last equation are equal to c_{vrst} by equation (11). Finally, we show that if v and w are distinct elements of $\Sigma(B)$, then

$$c_{vw} = c_v\cdot c_w.$$

Indeed, picking some s distinct from v and w, we have

$$c_v^2 = \frac{c_{vw}\cdot c_{vs}}{c_{ws}} \quad \text{and} \quad c_w^2 = \frac{c_{vw}\cdot c_{ws}}{c_{vs}},$$

so that $c_v^2 \cdot c_w^2 = c_{vw}^2$, as required. Finally, for S any subset of Σ_B of even cardinality, say $S = \{v_1, v_2, \ldots, v_{2n}\}$, we have

$$c_S = c_{v_1 v_2} \cdot c_{v_3 v_4} \cdots c_{v_{2n-1} v_{2n}} = \prod_{i=1}^{2n} c_{v_i},$$

by repeated application of equation (11). Then, for any B with $\Sigma_B \subset \Sigma(\pi)$, we get

$$\langle f_B, f_B \rangle = \frac{L(1, \mathrm{ad}^0 \pi)}{c_{\Sigma_B}} = \frac{L(1, \mathrm{ad}^0 \pi)}{\prod_{v \in \Sigma_B} c_v},$$

as required.

This section needs to be completed.

Appendices

A. Outline of student projects

A.1. **Project A: Periods on definite quaternion algebras.** In section 3, we outlined a proof of period relations for definite quaternion algebras over $\mathbf{Q}$ outside Eisenstein primes.

(1) Can you come up with a conjecture for what happens at Eisenstein primes, by computing some examples using Magma ?

Andrew Snowden, who will be assisting me with the course, has written some code that does some of these computations over $\mathbf{Q}$. This is a Magma program which takes as input a squarefree odd number N, computes all Brandt modules whose discriminant and level multiplies to N, groups the newforms in these Brandt modules according to how they correspond under Jacquet-Langlands and then computes the norms of these forms. This code along with instructions on how to use it is available at:

http://math.mit.edu/ asnowden/aws11/

Alternately, Appendix C by John Voight gives an outline of how to do such computations in specific cases in the more general case of totally real fields.

(2) State what you think should be the analogs of the results Prop. 3.2 and Prop. 3.3 when $\mathbf{Q}$ is replaced by a totally real field F. Can you find such results in the literature ? How far can you take the argument of Sec. 3 in the totally real case ?

(3) Using Magma again, compute examples over totally real fields to verify some cases of the period relations.

A.2. **Project B: Computing on Shimura curves I.** The goal of this project will be to compute Taylor expansions of modular forms on Shimura curves at CM points.

(1) We will start by putting ourselves in a very simple situation. Let K be the imaginary quadratic field $\mathbf{Q}(\sqrt{-7})$. This has class number one and the only units in the ring of integers are $\{\pm 1\}$. Let $B = M_2(\mathbf{Q})$ and E the elliptic curve $X_0(11)$. Since 11 is split in K, one can find a Heegner point on $X_0(11)$ coming from K. In fact, there are two such points corresponding to the two primes $\mathfrak{p}$, $\bar{\mathfrak{p}}$ in $\mathcal{O}_K$ such that $(11) = \mathfrak{p}\bar{\mathfrak{p}}$. Read the discussion in Gross-Zagier[7] Ch. II §1 and compute representatives τ_1, τ_2 for these Heegner points on the upper half plane. You can also compute this using SAGE, but if you've never seen this before it's a good idea to compute an example or two by hand. Here is sample code in SAGE to do this. (SAGE is somehow picking one of the two Heegner points, it's not clear to me how this choice is being made.)

```
sage: E=EllipticCurve('11a')
```

```
sage: P=E.heegner_point(-7,1)
sage: P.tau()
1/22*sqrt_minus_7 - 9/22
```

The arguments (-7,1) above indicate you want a Heegner point for $\mathbf{Q}(\sqrt{-7})$ of conductor 1.

(2) Let us pick $\tau = \tau_1$. Let f be the modular form corresponding to E. We would like to compute the Taylor expansion of f at τ. Of course, one can do this by just using the q-expansion of f. However, the point is to develop a technique to do this in a situation where q-expansions are not available. We will use the following formula (see [2] Ch. 5 and 6) which holds more generally:

$$[\delta^j f(\tau)]^2 = \frac{\mu^{\frac{k}{2}+j}(\mathfrak{n})}{w_{N,f}\cdot N^{\frac{k}{2}+j}} \frac{\Gamma(j+1)\Gamma(k+j)w_K\sqrt{|d_K|}\mathrm{Im}(\tau)^{-(k+2j)}}{(4\pi)^{k+2j+1}} \cdot L(k+j, f\times\theta_{\mu^j}), \tag{12}$$

where

- K is an imaginary quadratic field of odd discrminant d_K and class number 1.
- w_K is the number of roots of unity in K.
- f is a form of even weight k and odd conductor N, where all the primes dividing N are split in K. $w_{N,f}$ is the eigenvalue of the Atkin-Lehner operator w_N acting on f.
- $N = \mathfrak{n}\bar{\mathfrak{n}}$ and τ is a Heegner point on $\mathcal{H}$ corresponding to $\mathfrak{n}$ (well defined up to $\Gamma_0(N)$-action.)
- μ is the unique unramified Hecke character of K of infinity type $(2,0)$.
- $j \geq 0$ is an integer and $\delta^j f$ denotes the image of f under the Shimura-Maass operator applied j times.

Here is some sample code in Magma to compute the L-value above for the case $j = 0$. This uses the fact that f is the unique cusp form of weight 2 on $\Gamma_0(N)$ and θ_μ is the unique cusp form of weight 3 on $\Gamma_1(7)$.

```
> M:=ModularForms(11);
> M:=CuspidalSubspace(M); M;
Space of modular forms on Gamma_0(11) of weight 2 and dimension 1 over
Integer Ring.
> f:=M.1; f;
q - 2*q^2 - q^3 + 2*q^4 + q^5 + 2*q^6 - 2*q^7 - 2*q^9 - 2*q^10 + q^11
+ O(q^12)
> M:=ModularForms(Gamma1(7),3);
> M:=CuspidalSubspace(M);      M;
Space of modular forms on Gamma_1(7) of weight 3 and dimension 1 over
Integer Ring.
> g:=M.1; g;
q - 3*q^2 + 5*q^4 - 7*q^7 - 3*q^8 + 9*q^9 - 6*q^11 + O(q^12)
```

```
> Lf:=LSeries(f);
> Lg:=LSeries(g);
> L:=TensorProduct(Lf,Lg);
> Evaluate(L,2);
3.74353612485091521049660878267
```

For the more general case, one would need to write some code to find the local factors for the L-function of θ_{μ^j}. Also, Magma seems to slow down considerably as we increase j and even more so if we increase N, so we should try to find another way to evaluate the L-value above. One possibility is to use Rubinstein's lcalc in SAGE. (SAGE also has another L-value evaluator, namely Dokchitser's, but I couldn't get that to work.)

Now compute directly the value of $f(\tau)$ using the q-expansion and compare this with the value you got previously.

(3) Next repeat the same with $(\delta^j f(\tau))^2$ for the first few values of j. Again, compare this with what you get using the q-expansion. One check worth doing is the following: the ratio

$$\frac{(2\pi i)^{k+2j}\delta^j f(\tau)}{\Omega^{k+2j}}$$

should live in K, with Ω being a period of an elliptic curve over $\mathbf{Q}$ with CM by K. For example, one could take Ω to be the real period of the elliptic curve A of conductor $49A$ in Cremona's tables. You can compute Ω in Magma using:

```
> A:=EllipticCurve([1,-1,0,-2,-1]);
> RealPeriod(A);
1.93331170561681154673307683903
```

We now need to fix the sign of $\delta^j f(\tau)$. To do this, we will explore two different methods.

(4) The first possiblity is to guess the signs and use this to write down a power series expansion for f at τ. We can then evaluate this power series at $\gamma\tau$ for a large number of $\gamma \in \Gamma_0(N)$ and match this against the modularity of f:

$$f(\gamma\tau) = j(\gamma,\tau)^2 f(\tau).$$

Alternately, (as was suggested to me by John Voight) one could use the image of τ under Hecke operators T_p for many p, and check that the appropriate relations hold.

When expanding f in a power series at τ, one should be careful to use a good choice of coordinate. If one uses the usual coordinate on the upper half plane, then the power series will only converge on the maximal disc about τ in the upper half plane. So it's better to move to the unit disc model of the upper half plane via the change of variables:

$$z \mapsto \frac{z-\tau}{z-\bar{\tau}}.$$

Let F be the resulting function on the unit disc. Find a formula relating $\delta^j f(\tau)$ to the Taylor coefficients of F at 0. The power series for F at the origin will have radius of convergence 1.

(5) Here is another possible method to fix the signs. Find a formula analogous to equation (12) above for the value

$$\delta^{j_1} f(\tau) \cdot \overline{\delta^{j_2} f(\tau)},$$

with $j_1 \geq j_2 \geq 0$ nonnegative integers. Roughly, this should take the form

$$\delta^{j_1} f(\tau) \cdot \overline{\delta^{j_2} f(\tau)} = C(f, j_1, j_2, \tau) \cdot \langle \theta_{\chi_1}, D(f \cdot \theta_{\chi_2}) \rangle,$$

where

- χ_1 and χ_2 are certain Hecke characters of K of infinity type $k+j_1+j_2$ and j_1-j_2 respectively
- D is a certain differential operator taking forms of weight $k+j_1-j_2+1$ to forms of $k+j_1+j_2+1$.
- $C(f, j_1, j_2, \tau)$ is an explicit constant.

We will explore such a formula in Appendix B.

(6) Having computed the Taylor expansion of f, use it to compute the integral

$$\int_{\tau}^{\tau'} 2\pi i f(z) dz \tag{13}$$

where τ' is the Heegner point corresponding to $\bar{\mathfrak{p}}$. Let Λ be the period lattice for the elliptic curve E. Compute the point on E obtained as the image of the integral (13) in $\mathbf{C}/\Lambda$. You should get a K-rational point on E.

(7) Next, repeat all of the above for a newform f on a compact Shimura curve over $\mathbf{Q}$. For example, one could take B to be the indefinite quaternion algebra of discriminant 15 and f to correspond to an elliptic curve over $\mathbf{Q}$ of conductor 15. (There is a unique such curve up to isogeny.) Since 3 and 5 are inert in K, one can construct Heegner points corresponding to K on X_B. One thing you will need is a formula for the values of the Jacquet-Langlands transfer g of f at Heegner points analogous to equation (12) above. This should look like:

$$\frac{[\delta^j g(\tau)]^2}{\langle g, g\rangle / \langle f, f\rangle} = C \cdot \frac{\Gamma(j+1)\Gamma(k+j) w_K \sqrt{|d_K|} \mathrm{Im}(\tau)^{-(k+2j)}}{(4\pi)^{k+2j+1}} \cdot L(k+j, f \times \theta_{\mu^j}), \tag{14}$$

where

$$C = \frac{1}{w_{N,f}} \cdot \left(\frac{\overline{J(\mathfrak{j}, \tau)}}{J(\mathfrak{j}, \tau)} \cdot (\mu^{\rho} \cdot \mathbf{N}^{-1})(I) \right)^{\frac{k}{2}+j} \cdot \prod_{q | N^-} \frac{q-1}{q+1},$$

with $B = K + K\mathfrak{j}$ an orthogonal decomposition for the norm form and

$$I := \{\alpha \in K : \alpha\mathfrak{j} \in \mathcal{O}(N^-, N^+)\}.$$

(8) Finally, if we're successful with all of the above, we will think about the totally real case. One example that would be particularly interesting to compute is the following:
- F is a totally real cubic field.
- B is a quaternion algebra over F ramified at two infinite primes and nowhere else.
- E is an elliptic curve over F that is unramified everywhere. Then E can be realized as a quotient of the Shimura curve X_B.
- Let K be a CM field over F. Then X_B admits Heegner points associated to K.

Can one compute the images of such Heegner points in E ? Note that this is a situation in which methods coming from p-adic uniformization don't work since there are no finite primes at which B is ramified.

A.3. **Project C: Computing on Shimura curves II.** This goal of this project will be to pursue an alternative approach to computing modular forms on Shimura curves, suggested by Paul Nelson (who is one of the graduate students in our group). The idea is to compute directly the Shimizu lift. The method is described in detail in Appendix C.

B. Computing on Shimura Curves I: Expansions at CM points

The goal of this appendix is to outline one possible technique for computing with modular forms on Shimura curves. Since Shimura curves do not admit cusps, there are no q-expansions to work with. One alternative is to instead use the Taylor expansion at a well chosen CM point, in a suitably chosen coordinate. We will explain how this can be computed below.

B.1. **Basic setup.** We will put ourselves in the following very simple situation:

- N is a square-free positive integer.
- $N = N^+ \cdot N^-$ where N^- has an even number of prime factors.
- B is the indefinite quaternion algebra over $\mathbf{Q}$ with discriminant N^-. Fix an isomorphism $\iota : B \otimes \mathbf{R} \simeq M_2(\mathbf{R})$.
- $\mathcal{O}_{N^-,N^+}$ is an Eichler order of level N^+ in B, $U := \widehat{\mathcal{O}_{N^-,N^+}}^\times$.
- Γ is the group of norm 1 elements in M, thought of as a subgroup of $\mathbf{SL}_2(\mathbf{R})$ via the isomorphism ι.
- f is a newform on $\Gamma_0(N)$ of weight $2k$, trivial central character and coefficients in $\mathbf{Q}$.
- f_B denotes its Jacquet-Langlands transfer to $B^\times$. This can be viewed as a holomorphic function on $\mathcal{H}$, well defined at least up to scaling and satisfying

$$f_B(\gamma \cdot z) = J(\gamma, z)^{2k} f_B(z)$$

  for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. (Here we set $J(\gamma, z) := cz + d$, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\mathbf{GL}_2(\mathbf{R})$ and for future use, $j(\gamma, z) := \det(\gamma)^{-1/2} J(\gamma, z)$.)
- K is an imaginary quadratic field that is *Heegner* for f_B. i.e. K is split at all the primes dividing N^+, inert at all the primes dividing N^-. Thus K admits an embedding

$$\xi : K \hookrightarrow B$$

  such that

$$K \cap \mathcal{O}_{N^-,N^+} = \mathcal{O}_K.$$

- We assume that the only roots of unity in K are $\{\pm 1\}$ and that the class number of K is 1. Let μ denote the unique unramified Hecke character of K of infinity type $(2, 0)$, i.e. as a character on ideals,

$$\mu((a)) = a^2.$$

  Viewed as a character on $K^\times \setminus \mathbb{A}_K^\times$, it satisfies

$$\mu(x \cdot u \cdot a_\infty) = \mu(x) a_\infty^{-2}$$

  for all $u \in \widehat{\mathcal{O}_K}^\times$ and $a_\infty \in K_\infty^\times$. Note that for every positive integer $2\ell \geq 2$, the character μ^ℓ is the unique unramified Hecke character of K of infinity type $(2\ell, 0)$.

- We denote by δ_ℓ the Shimura-Maass operator on C^∞-forms of weight ℓ for Γ by
$$\delta_\ell(g) = \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{\ell}{2iy}\right) g.$$
This maps C^∞-forms of weight ℓ to C^∞-forms of weight $\ell+2$. Also, we will denote by $\delta_\ell^{(j)}$ the composite
$$\delta_\ell^{(j)}(g) = \delta_{\ell+2j-2} \circ \cdots \circ \delta_\ell,$$
which takes forms of weight t to forms of weight $t+2j$. We will often just write $\delta^j g$ for $\delta_\ell^{(j)}(g)$.
- Let us fix an embedding ξ as above. The image of $K^\times$ in $\mathbf{GL}_2(\mathbf{R})$ under $\iota \circ \xi$ has a unique fixed point τ on $\mathcal{H}$. We would like to expand f_B as a power series at τ. Obviously, this is equivalent to computing $\delta^j f_B(\tau)$ for all $j \geq 0$.

Let F_B^j be the automorphic form on $B_{\mathbb{A}}^\times$ defined as follows. For any $x \in B_{\mathbb{A}}^\times$, write (using strong approximation)

$$x = \gamma \cdot (u \cdot \gamma_\infty) \tag{15}$$

where γ is in $B^\times$, u is in U and $\gamma_\infty \in (B \otimes \mathbf{R})^+ = \mathbf{GL}_2(\mathbf{R})^+$. Then set
$$F_B^j(x) = j(\gamma_\infty, \tau)^{-(2k+2j)} \delta^j f_B(\gamma_\infty \tau).$$
This is easily checked to be independent of the decomposition (15).

Lemma B.1. *Suppose $\alpha \in K_\infty^\times$. Then*
$$F_B^j(x\alpha) = F_B^j(x) \cdot \alpha^{-(k+j)} \overline{\alpha}^{k+j}.$$

Now define a_j by
$$a_j := \int_{\mathbb{A}_K^\times} F_B^j(x) \cdot (\mu^{-(k+j)} \mathbf{N}_K^{k+j})(x) d^\times x$$

Lemma B.2.
$$a_j = \delta^j f_B(\tau).$$

We will give for every pair (j_1, j_2) a formula for $\overline{a_{j_1}} a_{j_2}$. Without loss of generality we may assume $j_1 \geq j_2$.

B.2. **Maass operators and Whittaker coefficients.** In this section we will work on $\mathbf{GL}_2$ and compute the Whittaker coefficients of $\delta^j f$ where f is a normalized eigenform of weight ℓ on $\Gamma_0(N)$. Let us write F for the adelic form associated to f and F^j for the adelic form associated to $\delta^j f$.

Suppose
$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

Then

$$F\left(\left[\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right]_\infty\right) = f(iy)y^{\ell/2} = \sum_{n=1}^{\infty} a_n y^{\ell/2} e^{-2\pi n y}.$$

On the other hand, defining

$$W_F(g) = \int_{\mathbf{Q}\backslash\mathbb{A}} F\left(\left(\begin{array}{cc} 1 & x \\ 0 & 1 \end{array}\right) g\right) \overline{\tau(x)} dx,$$

we have

$$F\left(\left[\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right]_\infty\right) = \sum_{\xi\in\mathbf{Q}^\times} W_F\left(\begin{array}{cc} \xi y & 0 \\ 0 & 1 \end{array}\right) = \sum_{\xi\in\mathbf{Q}^\times} W_{F,\mathrm{fin}}\left(\left[\begin{array}{cc} \xi & 0 \\ 0 & 1 \end{array}\right]\right) W_{F,\infty}\left(\left[\begin{array}{cc} \xi y & 0 \\ 0 & 1 \end{array}\right]\right),$$

where $W_{F,\mathrm{fin}}$ and $W_{F,\infty}$ are normalized by

$$W_{F,\mathrm{fin}}(\mathbf{1}) = 1, \quad W_{F,\infty}\left(\left[\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right]\right) = y^{\ell/2} e^{-2\pi y} \mathbf{I}_{\mathbf{R}^+}(y).$$

Now one computes:

$$\delta f(z) = \sum_{n=1}^{\infty} \left(n - \frac{\ell}{4\pi y}\right) a_n e^{2\pi i n z};$$

$$\delta^2 f(z) = \sum_{n=1}^{\infty} \left(n^2 - \frac{2\ell+2}{4\pi y} n + \frac{\ell^2+\ell}{(4\pi y)^2}\right) a_n e^{2\pi i n z},$$

and so on. i.e.

Proposition B.3. *There is a monic polynomial $p_j(t)$ of degree j with integer coefficients such that*

$$W_{F^j}\left(\left[\begin{array}{cc} y & 0 \\ 0 & 1 \end{array}\right]_\infty\right) = \frac{1}{(4\pi)^j} p_j(4\pi y) y^{\ell/2} e^{-2\pi y}.$$

Example B.4. We have $p_1(t) = t - \ell$ and $p_2(t) = t^2 - (2\ell+2)t + (\ell^2+\ell)$.

B.3. **Explicit theta lifts.** Let ψ denote the additive character of $\mathbb{A}/\mathbf{Q}$ given by $\psi((x_v)_v) = \prod_v \psi_v(x_v)$, where

$$\psi_\infty(x) = e^{2\pi i x}, \qquad \psi_q(x) = e^{-2\pi i x} \qquad \text{for} \quad x \in \mathbf{Z}\left[\frac{1}{q}\right] \subset \mathbf{Q}_q.$$

Let $(V, \langle,\rangle)$ be an orthogonal space over $\mathbf{Q}$, and denote by $\mathbf{O}(V)$ (resp. $\mathbf{GO}(V)$) its isometry (resp. similitude) group. Recall the Weil representation $r_\psi = \prod_v r_{\psi,v}$ of the group $\mathbf{SL}_2(\mathbb{A}) \times \mathbf{O}(V)(\mathbb{A})$ on the Schwartz space $\mathcal{S}(V(\mathbb{A}))$. On the orthogonal group, $r_{\psi,v}$ is given by

$$r_{\psi,v}(g)\varphi(x) = \varphi(g^{-1}\cdot x) \qquad \text{for } g \in \mathbf{O}(V), \varphi \in \mathcal{S}(V(\mathbf{Q}_v)).$$

On $\mathbf{SL}_2(\mathbf{Q}_v)$, the representation $r_{\psi,v}$ is described by its action on the matrices

$$U(a) := \left(\begin{array}{cc} 1 & a \\ 0 & 1 \end{array}\right), \quad D(a) := \left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right), \quad W = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),$$

by the equations

$$\begin{aligned} r_{\psi,v}(U(a))\varphi(x) &= \psi_v(\frac{1}{2}\langle ax,x\rangle)\varphi(x), \\ r_{\psi,v}(D(a))\varphi(x) &= (a,V)_v|a|_v^{\dim(V)/2}\varphi(ax), \\ r_{\psi,v}(W)\varphi(x) &= \gamma_V\hat{\varphi}(x), \end{aligned}$$

where

- $(\cdot,\cdot)_v$ denotes the Hilbert symbol so that $(\cdot,V)_v$ is the quadratic character associated to V.
- γ_V is an eighth root of unity, the exact value of which can be found in [15] §1.
- The Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(x) = \int_{V(\mathbf{Q}_v)} \varphi(y)\psi_v(\langle y,x\rangle)dy,$$

  the measure dy on $V(\mathbf{Q}_v)$ being chosen such that $\hat{\hat{\varphi}}(x) = \varphi(-x)$.

We will need to extend the Weil representation to similitude groups, following Harris-Kudla [9]. Let $\mathcal{R}$ be the group defined by:

$$\mathcal{R} := \{(g,h) \in \mathbf{GL}_2 \times \mathbf{GO}(V) : \det(g) = \nu(h)\}$$

where ν denotes the similitude character of $\mathbf{GO}(V)$. Then r_ψ can be extended to $\mathcal{R}(\mathbb{A})$ by

$$r_\psi(g,h)\varphi = r_\psi\left(g\cdot\begin{pmatrix} 1 & 0 \\ 0 & \det g^{-1}\end{pmatrix}\right)L(h)\varphi,$$

where

$$L(h)\varphi(x) = |\nu(h)|^{-\dim(V)/4}\varphi(h^{-1}x).$$

Let $\mathbf{GO}(V)^0$ denote the algebraic connected component of $\mathbf{GO}(V)$. If F is an automorphic form on $\mathbf{GL}_2(\mathbb{A})$ and $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we define for $h \in \mathbf{GO}(V)(\mathbb{A})$,

$$\theta_\varphi(F)(h) := \int_{\mathbf{SL}_2(\mathbf{Q})\backslash\mathbf{SL}_2(\mathbb{A})} \sum_{x\in V(\mathbf{Q})} r_\psi(gg',h)\varphi(x)F(gg')d^{(1)}g,$$

where g' is chosen such that $\det(g') = \nu(h)$. Likewise, in the opposite direction, if F' is an automorphic form on $\mathbf{GO}(V)^0(\mathbb{A})$, and $g \in \mathbf{GL}_2(\mathbb{A})$ is such that $\det(g) \in \nu(\mathbf{GO}(V)(\mathbb{A}))$, we set

$$\theta^t_\varphi(F')(g) := \int_{\mathbf{O}(V)(\mathbf{Q})\backslash\mathbf{O}(V)(\mathbb{A})} \sum_{x\in V(\mathbf{Q})} r_\psi(g,hh')\varphi(x)F'(hh')dh,$$

where $h' \in \mathbf{GO}(V)^0(\mathbb{A})$ is chosen such that $\det(g) = \nu(h')$. (We refer the reader to [22], §1, for the choices of measures in the above and in what follows.) If π (resp. Π) is an automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ (resp. of $\mathbf{GO}(V)^0(\mathbb{A})$), we define

$$\theta(\pi) := \{\theta_\varphi(F) : F \in \pi, \varphi \in \mathcal{S}(V(\mathbb{A}))\};$$
$$\theta^t(\Pi) := \{\theta^t_\varphi(F') : F' \in \Pi, \varphi \in \mathcal{S}(V(\mathbb{A}))\}.$$

Now set $V = B$ and consider V as an orthogonal space over $\mathbf{Q}$ with bilinear form $\langle x, y\rangle = \frac{1}{2}(xy^i + yx^i)$, where $x \mapsto x^i$ denotes the main involution. The associated quadratic form is just $x \mapsto xx^i = \nu(x)$, where $\nu(\cdot)$ is the reduced norm. The group $\mathbf{GO}(V)^0$ is identified with $\mathbf{Q}^\times \backslash B^\times \times B^\times$ via the map $(\alpha, \beta) \mapsto \delta(\alpha, \beta)$ where $\delta(\alpha,\beta)(x) = \alpha x \beta^{-1}$. Thus an automorphic representation of $\mathbf{GO}(V)^0(\mathbb{A})$ is identified with a pair (π_1, π_2) of representations of $B_{\mathbb{A}}^\times$, such that the product of the central characters of π_1 and π_2 is trivial.

Let π denote the (unitary) automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ associated to f. The following theorem is the classical Jacquet-Langlands correspondence realized using theta functions, and is essentially due to Shimizu [32]. (See also [38] §3.2.)

Theorem B.5. *(1) $\theta(\bar{\pi}) = \bar{\pi} \times \pi$, where $\bar{\pi} = \pi^\vee \simeq \pi$, since π has trivial central character.*
(2) $\theta^t(\pi \times \bar{\pi}) = \pi$.

We will need a statement involving specific forms in π and $\bar{\pi}$ and explicit theta functions i.e. explicit choices of Schwartz functions. Recall that we have picked an embedding $K \hookrightarrow B$, so we can write $B = K + K^\perp = K + K\mathfrak{j}$ where $\mathrm{tr}(\mathfrak{j}) = 0$ and

$$\mathfrak{j}x = \bar{x}\mathfrak{j} \quad \text{for} \quad x \in K.$$

We consider the following Schwartz function: $\varphi := \otimes_q \varphi_q$ where

(i) For q finite, $\varphi_q := \mathbf{I}_{\mathcal{O}_{N^-,N^+} \otimes \mathbf{Z}_q}$ i.e. the characteristic function of $\mathcal{O}_{N^-,N^+} \otimes \mathbf{Z}_q$.

(ii) For $q = \infty$, we identify $M_2(\mathbf{R}) = (K \otimes \mathbf{R}) + (K \otimes \mathbf{R})^\perp = \mathbf{C} + \mathbf{C}\mathfrak{j}$ and define

$$\varphi_\infty^\Xi(\mathbf{u} + \mathbf{v}\mathfrak{j}) = \bar{\mathbf{u}}^{m_1}\bar{\mathbf{v}}^{m_2} e^{-2\pi(|\langle \mathbf{u},\mathbf{u}\rangle| + |\langle \mathbf{v}\mathfrak{j},\mathbf{v}\mathfrak{j}\rangle|)} = \bar{\mathbf{u}}^{m_1}\bar{\mathbf{v}}^{m_2} e^{-2\pi(|\mathbf{u}|^2 + |\mathrm{N}\mathfrak{j}||\mathbf{v}|^2)},$$

for $\mathbf{u}, \mathbf{v} \in \mathbf{C}$, where

$$m_1 = 2k + j_1 + j_2 \quad \text{and} \quad m_2 = j_1 - j_2.$$

Lemma B.6. *Suppose* $\kappa_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbf{SO}_2(\mathbf{R})$ *and* $\kappa_1, \kappa_2 \in (K \otimes \mathbf{R})^{(1)} \subset \mathbf{GL}_2(\mathbf{R})$. *Then*

$$r_\psi(\kappa_\theta, (\kappa_1, \kappa_2))\varphi_\infty = e^{-i(2k+2j_2)\theta} \cdot \kappa_1^{2k+2j_1} \cdot \kappa_2^{-(2k+2j_2)} \varphi_\infty.$$

Proof. □

Lemma B.7. *Let q be a finite prime and suppose $\alpha \in U_q$ and $\beta, \gamma \in U_q'$, are such that*

$$\det(\alpha) = \nu(\beta) \cdot \nu(\gamma)^{-1},$$

so that $(\alpha, (\beta, \gamma))$ may be viewed as an element of $\mathcal{R}(\mathbf{Q}_q)$. Then

$$r_\psi(\alpha, (\beta,\gamma))\varphi_q = \varphi_q.$$

Proposition B.8.

$$\theta_\varphi^t(F_B^{j_1} \times \overline{F_B^{j_2}}) = C_1 \cdot \langle F_B^{j_2}, F_B^{j_2}\rangle \cdot F^{j_2},$$

where

$$C_1 = (-1)^{j_2} 2^{-2j_1-1} \cdot \frac{j_1!(2k+j_1)!}{j_2!(2k+j_2)!} \cdot \pi^{-j_1-1} \cdot (-4i\pi\Lambda)^{j_2-j_1} \cdot \mathrm{vol}(U'^{(1)}). \tag{16}$$

Proof. Let $F' := \theta_\varphi^t(F_B^{j_1} \times \overline{F_B^{j_2}})$. We first show that $F' = C_1' \cdot F^{j_2}$ for some constant C_1'. Note that for $u \in U$ and $\kappa \in \mathbf{SO}_2(\mathbf{R})$, by Lemmas B.6 and B.7,

$$F'(gu\kappa_\theta) = \int \sum_x r_\psi(gu\kappa, h \cdot (u,1))\varphi(x)(F_B^{j_1} \times \overline{F_B^{j_2}})(h \cdot (u,1))d^{(1)}h = e^{-i(2k+2j_2)\theta} F'(g)$$

Since $\theta^t(\pi \otimes \bar{\pi}) = \pi$, it follows by Casselman's theorem that $F' = C_1' \cdot F^{j_2}$ for some scalar C_1'. Clearly, C_1' is just the first Fourier coefficient of F'. To find the value of C_1', one computes the Whittaker coefficients of F'. As in [38] Sec. 3.2.1,

$$W_{F',\psi}(g) = \frac{1}{2} \int_{B_{\mathbf{Q}}^\times \backslash B_{\mathbb{A}}^\times} \Psi(g,\beta) \overline{F_B^{j_2}}(\beta) d^\times \beta,$$

where

$$\Psi(g,\beta) = \int_{B_{\mathbb{A}}^{\det(g)}} r_\psi(g, \delta(\alpha,1))\varphi(1) F_B^{j_1}(\beta\alpha) d^{(1)}\alpha.$$

We specialize to $g_0 = \mathbf{1}_f \cdot \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}_\infty$ with $y > 0$. Then

$$\begin{aligned}
\Psi(g_0,\beta) &= \int_{B_{\mathbb{A}}^{(1)}} r_\psi(\mathbf{1}_f \cdot \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}_\infty, \delta(\alpha,1) \cdot \delta(\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y} \end{pmatrix}_\infty, 1))\varphi(1) \\
&\qquad \cdot F_B^{j_1}(\beta\alpha \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y} \end{pmatrix}_\infty) d^{(1)}\alpha \\
&= y \int_{B_{\mathbb{A}}^{(1)}} \varphi((\sqrt{y})_\infty \alpha^{-1}) F_B^{j_1}(\beta\alpha) d^{(1)}\alpha.
\end{aligned}$$

This integral can be computed one place at a time since both $F_B^{j_1}$ and φ are pure tensors. We first consider finite primes q. In this case, if $\varphi_q(\alpha_q^{-1}) \neq 0$, then $\alpha_q^{-1} \in U_q'$. Hence $\alpha_q \in U_q'$ as well and $F_B^{j_1}(\beta\alpha_q) = F_B^{j_1}(\beta)$. Thus

$$\int_{B^{(1)}(\mathbf{Q}_q)} \varphi_q(\alpha_q^{-1}) F_B^{j_1}(\beta\alpha_q) d^{(1)}\alpha_q = \mathrm{vol}(U_q'^{(1)}) \cdot F_B^{j_1}(\beta).$$

Next, we compute the local integral at ∞. Note first that for $\kappa \in K_\infty^{(1)}$, we have

$$\begin{aligned}\Psi(g_0,\beta\kappa) &= y\int_{B_{\mathbb{A}}^{(1)}} \varphi((\sqrt{y})_\infty\alpha^{-1})F_B^{j_1}(\beta\kappa\alpha)d^{(1)}\alpha \\ &= y\int_{B_{\mathbb{A}}^{(1)}} \varphi((\sqrt{y})_\infty\alpha^{-1}\kappa^{-1})F_B^{j_1}(\beta\alpha)d^{(1)}\alpha \\ &= \kappa^{m_2-m_1}\cdot y\int_{B_{\mathbb{A}}^{(1)}} \varphi((\sqrt{y})_\infty\alpha^{-1})F_B^{j_1}(\beta\alpha)d^{(1)}\alpha = \kappa^{-(2k+2j_2)}\Psi(g_0,\beta).\end{aligned}$$

This shows that $\Psi(g_0,\beta)$ is a scalar multiple of $F_B^{j_2}$. To find the value of this scalar, we we use that $\mathbf{SL}_2(\mathbf{R})$ is conjugate to $\mathbf{SU}(1,1)$ in $\mathbf{SL}_2(\mathbf{C})$ and that a model for π_{2k}^+ is provided by analytic functions on the unit disc. Let us pick $\gamma \in \mathbf{SL}_2(\mathbf{C})$ such that

$$\gamma^{-1}\mathbf{SL}_2(\mathbf{R})\gamma = \mathbf{SU}(1,1),$$

and further satisfying

$$\gamma^{-1}\alpha\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix},$$

for $\alpha \in (K\otimes\mathbf{R})^\times$. (As usual we think of $(K\otimes\mathbf{R})^\times$ as sitting inside $\mathbf{SL}_2(\mathbf{R})$ via $\iota\circ\xi$.) Then we define a scalar $\Lambda \in K$ by

$$\gamma^{-1}\mathfrak{j}\gamma = \Lambda\cdot\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If ζ is the variable on the disc, then the action of $\mathbf{SU}(1,1)$ is given by

$$\begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \bar{\boldsymbol{\beta}} & \bar{\boldsymbol{\alpha}} \end{pmatrix}\cdot h(\zeta) = (-\boldsymbol{\beta}\zeta+\bar{\boldsymbol{\alpha}})^{-2k}h\left(\frac{\boldsymbol{\alpha}\zeta-\bar{\boldsymbol{\beta}}}{-\boldsymbol{\beta}\zeta+\bar{\boldsymbol{\alpha}}}\right).$$

In this model, the function $h_j : \zeta \mapsto \zeta^j$ has weight $2k+2j$. Write

$$\begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \bar{\boldsymbol{\beta}} & \bar{\boldsymbol{\alpha}} \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & a \end{pmatrix}\cdot\begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix},$$

where $a = \cosh t/2$ and $b = \sinh t/2$. In these variables the measure on $\mathbf{SU}(1,1)$ is $d\mu = \frac{1}{8\pi^2}\sinh t dt d\phi d\psi$.Then

$$\begin{aligned} y\int_{B_{\mathbf{R}}^{(1)}} \varphi((\sqrt{y})_\infty\alpha^{-1})\pi_{2k}^+(\alpha)h_{j_1}d^{(1)}\alpha(\zeta) &= \zeta^{j_1}\cdot y^{1+k+j_1}\cdot \\ &\int e^{i\phi(2j_1-2j_2)}a^{j_1+j_2}(\Lambda^{-1}b)^{j_1-j_2}e^{-2\pi y(a^2+b^2)}\cdot(1-\frac{b}{a}e^{-2i\phi}\zeta^{-1})^{j_1}\cdot(1-\frac{b}{a}e^{2i\phi}\zeta)^{-2k-j_1}d\mu.\end{aligned}$$

Suppose

$$(1-X)^{j_1} = \sum_{\ell\geq 0} c_\ell X^\ell$$

and

$$(1-X)^{-2k-j_1} = \sum_{\ell \geq 0} d_\ell X^\ell.$$

Then

$$\begin{aligned}
& y \int_{B_{\mathbf{R}}^{(1)}} \varphi((\sqrt{y})_\infty \alpha^{-1}) \pi_{2k}^{+}(\alpha) h_{j_1}(\zeta) \\
= \; & \zeta^{j_2} y^{k+j_1+1} \Lambda^{j_2-j_1} \cdot \left(\sum_{\substack{\ell_1,\ell_2 \geq 0 \\ \ell_1-\ell_2=j_1-j_2}} c_{\ell_1} d_{\ell_2} (b/a)^{\ell_1+\ell_2} \right) \cdot a^{j_1+j_2} b^{j_1-j_2} e^{-2\pi y(a^2+b^2)} \cdot \sinh t dt \\
= \; & \zeta^{j_2} y^{k+j_1+1} \Lambda^{j_2-j_1} \cdot \int \sum_{0 \leq r \leq j_2} c_{r+(j_1-j_2)} d_r (b/a)^{2r+(j_1-j_2)} \cdot a^{j_1+j_2} b^{j_1-j_2} e^{-2\pi y(a^2+b^2)} \cdot \sinh t dt \\
= \; & 2^{-j_1} \zeta^{j_2} y^{k+j_1+1} \Lambda^{j_2-j_1} \cdot \sum_{0 \leq r \leq j_2} c_{r+(j_1-j_2)} d_r \cdot \int_1^\infty (t+1)^{j_2-r} (t-1)^{r+j_1-j_2} e^{-2\pi y(t-1)} dt \\
= \; & h_{j_2}(\zeta) y^{k+j_1+1} \Lambda^{j_2-j_1} \cdot 2^{-j_1} \cdot \sum_{0 \leq r \leq j_2} c_{r+(j_1-j_2)} d_r \cdot e^{-2\pi y} \int_0^\infty (t+2)^{j_2-r} t^{r+j_1-j_2} e^{-2\pi y t} dt \\
= \; & h_{j_2}(\zeta) y^{k+j_1+1} \Lambda^{j_2-j_1} \cdot 2^{-j_1} e^{-2\pi y} \cdot \\
& \qquad \sum_{\substack{0 \leq r \leq j_2 \\ 0 \leq i \leq j_2-r}} c_{r+(j_1-j_2)} d_r \int_0^\infty \binom{j_2-r}{i} t^{i+r+j_1-j_2} 2^{j_2-r-i} e^{-2\pi y t} dt \\
= \; & h_{j_2}(\zeta) \Lambda^{j_2-j_1} \cdot e^{-2\pi y} \cdot I(y)
\end{aligned}$$

where

$$\begin{aligned}
I(y) \; = \; & 2^{-j_1} y^{k+j_1+1} \sum_{\substack{0 \leq r \leq j_2 \\ 0 \leq i \leq j_2-r}} (-1)^{j_2-r} 2^{j_2-i-r} \binom{j_1}{j_2-r} \binom{2k+j_1-1+r}{r} \binom{j_2-r}{i} \cdot \\
& \qquad (2\pi y)^{-(i+r+j_1-j_2+1)} \Gamma(i+r+j_1-j_2+1) \\
= \; & 2^{-j_1} (2\pi)^{-j_1-1} y^k \cdot \\
& \sum_{\substack{0 \leq r \leq j_2 \\ 0 \leq i \leq j_2-r}} \left[(-1)^{j_2-r} \binom{j_1}{j_2-r} \binom{2k+j_1-1+r}{r} \binom{j_2-r}{i} \Gamma(i+r+j_1-j_2+1) \right] (4\pi y)^{j_2-i-r} \\
= \; & y^k q_j(4\pi y),
\end{aligned}$$

for a polynomial q_j. The leading coefficient of q_j is easy to compute, since it corresponds to the case $i = r = 0$. It is

$$c := 2^{-j_1} (2\pi)^{-j_1-1} \cdot (-1)^{j_2} \binom{j_1}{j_2} \Gamma(j_1-j_2+1) = 2^{-2j_1-1} \cdot (-1)^{j_2} \pi^{-j_1-1} \cdot \frac{j_1!}{j_2!}.$$

Thus it follows (though this is not directly evident from the formula above) that

$$I(y) = c \cdot y^k p_{j_2}(4\pi y).$$

Finally, we note that if F_B corresponds to h_1, then $R^j(F_B)$ corresponds to

$$2k(2k+1)\cdots(2k+j)\cdot i^j h_j,$$

hence F_B^j corresponds to

$$\frac{1}{(4\pi)^j}\cdot 2k(2k+1)\cdots(2k+j)i^j h_j.$$

Thus if $F_B^{j_1}$ corresponds to h_{j_1}, then h_{j_2} corresponds to

$$(4\pi)^{j_2-j_1}(2k+j_2+1)\cdots(2k+j_1)i^{j_1-j_2}F_B^{j_2}.$$

Combining the local computations, we find

$$\Psi(g_0,\beta) = c(-4i\pi\Lambda)^{j_2-j_1}\cdot(2k+j_2+1)\cdots(2k+j_1)\cdot \mathrm{vol}(U'^{(1)})\cdot y^k p_{j_2}(4\pi y)e^{-2\pi y}\cdot F_B^{j_2}(\beta).$$

Hence

$$W_{F',\psi}(g_0) = c(-4i\pi\Lambda)^{j_2-j_1}\cdot(2k+j_2+1)\cdots(2k+j_1)\,\mathrm{vol}(U'^{(1)})\cdot y^k p_{j_2}(4\pi y)\langle F_B^{j_2},F_B^{j_2}\rangle,$$

and

$$F' = (4\pi)^{j_2}\cdot c(-4i\pi\Lambda)^{j_2-j_1}\cdot(2k+j_2+1)\cdots(2k+j_1)\cdot\mathrm{vol}(U'^{(1)})\cdot\langle F_B^{j_2},F_B^{j_2}\rangle\cdot F^{j_2}.$$

Thus $F' = C_1\cdot\langle F_B^{j_2},F_B^{j_2}\rangle\cdot F^{j_2}$, where

$$C_1 = (-1)^{j_2}2^{2j_2-2j_1-1}\cdot\frac{j_1!(2k+j_1)!}{j_2!(2k+j_2)!}\cdot\pi^{j_2-j_1-1}\cdot(-4i\pi\Lambda)^{j_2-j_1}\cdot\mathrm{vol}(U'^{(1)}).$$

□

Remark B.9.

$$\mathrm{vol}(U'^{(1)}) = \zeta(2)^{-1}\cdot\prod_{q|N^+}\frac{1}{q+1}\cdot\prod_{q|N^-}\frac{1}{q-1}.$$

Proposition B.10.

$$\theta_\varphi(\overline{F^{j_2}}) = C_2\cdot(\overline{F_B^{j_1}}\times F_B^{j_2}),$$

where

$$C_2 = C_1\cdot\mathrm{Im}(\tau)^{2k+2j_1}\cdot\langle F,F\rangle/\langle F_B,F_B\rangle.$$

Proof. By a calculation as in (17) above and another application of Casselman's theorem, we have $\theta_\varphi(\overline{F^{j_2}}) = C_2\cdot(\overline{F_B^{j_1}}\times F_B^{j_2})$ for some constant C_2. To compute C_2, one studies the theta lift in the opposite direction and uses the seesaw principle. Indeed, the seesaw principle and Proposition B.8 imply that

$$C_2\langle F_B^{j_1},F_B^{j_1}\rangle\langle F_B^{j_2},F_B^{j_2}\rangle = \langle\theta_\varphi(\overline{F^j}),\overline{F_B^{j_1}}\times F_B^{j_2}\rangle = \langle\overline{F^j},\overline{\theta_\varphi^t(F_B^{j_1}\times\overline{F_B^{j_2}})}\rangle = C_1\langle F^j,F^j\rangle\langle F_B^{j_2},F_B^{j_2}\rangle.$$

i.e.

$$C_2 = C_1 \cdot \langle F^j, F^j\rangle / \langle F_B^{j_1}, F_B^{j_1}\rangle.$$

But

$$\langle F^{j_1}, F^{j_1}\rangle / \langle F_B^{j_1}, F_B^{j_1}\rangle = \mathrm{Im}(\tau)^{2k+2j_1} \cdot \langle F, F\rangle / \langle F_B, F_B\rangle,$$

whence

$$C_2 = C_1 \cdot \mathrm{Im}(\tau)^{2k+2j_1} \cdot \langle F, F\rangle / \langle F_B, F_B\rangle. \tag{17}$$

□

B.4. **An application of seesaw duality.** Let $V_1 = K$ (viewed as a subspace of V) and $V_2 = V_1^{\perp}$. Then

$$\mathbf{GO}(V_1)^0 \simeq \mathbf{GO}(V_2)^0 \simeq K^{\times},$$
$$\mathbf{H} := \mathbf{G}(\mathbf{O}(V_1) \times \mathbf{O}(V_2))^0 = \mathbf{G}(K^{\times} \times K^{\times}),$$

and via this identification the map $\delta : K^{\times} \times K^{\times} \to \mathbf{H}$ is

$$\delta(\alpha, \beta) = (\alpha\beta^{-1}, \alpha(\beta^{\rho})^{-1}).$$

Since $\mu \cdot \mathbf{N}_K^{-1}$ has trivial central character (i.e. its restriction to $\mathbb{A}_{\mathbf{Q}}^{\times}$ is trivial), there exists a Hecke character η of K of infinity type $(1,0)$, such that

$$\mu \cdot \mathbf{N}_K^{-1} = \eta \cdot (\eta^{\rho})^{-1},$$

where $\eta^{\rho}(x) := \eta(\bar{x})$. Set

$$\chi_1 := (\overline{\eta^{\rho}})^{2k+j_1+j_2} \qquad \chi_2 := \eta^{k+j_1}(\overline{\eta^{\rho}})^{-(k+j_2)} = \eta^{k+j_1}(\eta^{\rho})^{k+j_2}\mathbf{N}_K^{-(k+j_2)},$$
$$\omega_1 := \chi_1 \cdot \mathbf{N}_K^{-\frac{1}{2}(2k+j_1+j_2)}, \qquad \omega_2 := \chi_2 \cdot \mathbf{N}_K^{-\frac{1}{2}(j_1-j_2)}.$$

Then

$$\omega_1 \cdot \omega_2 = (\overline{\eta^{\rho}})^{k+j_1}\eta^{k+j_1}\mathbf{N}_K^{-(k+j_1)} = (\overline{\eta^{\rho}})^{k+j_1}(\bar{\eta})^{-(k+j_1)} = (\bar{\mu})^{-(k+j_1)}\mathbf{N}_K^{k+j_1},$$
$$\omega_1 \cdot \omega_2^{\rho} = \eta^{k+j_2}(\eta^{\rho})^{-(k+j_2)} = \mu^{k+j_2} \cdot \mathbf{N}_K^{-(k+j_2)},$$

and

$$\begin{aligned} \omega_1(xy^{-1}) \cdot \omega_2(x\bar{y}^{-1}) &= (\omega_1\omega_2)(x) \cdot (\omega_1\omega_2^{\rho})^{-1}(y) \\ &= (\bar{\mu})^{-(k+j_1)}\mathbf{N}_K^{k+j_1}(x) \cdot (\mu^{-(k+j_2)}\mathbf{N}_K^{k+j_2})(y). \end{aligned}$$

Thus

$$\begin{aligned} \overline{a_{j_1}} \cdot a_{j_2} &= \int_{K^{\times}\backslash\mathbb{A}_K^{\times} \times K^{\times}\backslash\mathbb{A}_K^{\times}} (\overline{F_B^{j_1}} \times F_B^{j_2})(x,y) \cdot \\ &\qquad (\bar{\mu})^{-(k+j_1)}\mathbf{N}_K^{k+j_1}(x) \cdot (\mu^{-(k+j_2)}\mathbf{N}_K^{k+j_2})(y) d^{\times}x d^{\times}y \\ &= \frac{1}{C_2}\int_{\mathbf{H}(\mathbf{Q})\backslash H(\mathbb{A})} \theta_{\varphi}(\overline{F^{j_2}})(h) \cdot (\omega_1, \omega_2)(h) dh \\ &= \frac{1}{C_2}\int_{\mathbf{GL}_2(\mathbf{Q})\backslash\mathbf{GL}_2(\mathbb{A})} \overline{F^{j_2}}(g)\theta_{\varphi}^{t}(\omega_1, \omega_2)(g) dg \end{aligned}$$

We now show that $\theta^t_\varphi(\omega_1,\omega_2)$, which is ostensibly a *sum* of products of theta functions associated to ω_1 and ω_2 (with different choices of Schwartz function) may in fact be replaced by just a single product of two theta functions. Notice that the Schwartz function φ_v splits as a pure tensor in $\mathcal{S}(V_1(\mathbf{Q}_v))\otimes\mathcal{S}(V_2(\mathbf{Q}_v))$ for $v=\infty$ and for all finite primes v such that v is unramified in K. For ramified primes this is no longer the case, however we shall show below that at such primes we can replace this sum of pure tensors by a different pure tensor in computing the integral above.

We work locally now: suppose $K=\mathbf{Q}_p(\zeta)$, where p is an odd prime, $\zeta^2=\pi$, with π being a uniformizer in $\mathbf{Q}_p$. We may assume $B=M_2(\mathbf{Q}_p)$, $\mathcal{O}_B=M_2(\mathbf{Z}_p)$, the embedding ξ is given by

$$a+b\zeta\mapsto\begin{pmatrix} a & b\\ b\pi & a\end{pmatrix}$$

and $\mathsf{j}=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$. Then

$$\varphi_p=\sum_{i=0}^{p-1}\varsigma_i\otimes\vartheta_i$$

where

$$\varsigma_i=\mathbf{I}_{\mathbf{Z}_p+(\frac{i}{\pi}+\mathbf{Z}_p)\zeta}\qquad \vartheta_j=\mathbf{I}_{[\mathbf{Z}_p+(\frac{j}{\pi}+\mathbf{Z}_p)\zeta]\mathsf{j}}.$$

We write $I(\varsigma_i,\vartheta_j)$ for the integral above where we fix the Schwartz function outside p and take $\varsigma_i\otimes\vartheta_j$ as the Schwartz function at p. Then

$$I=\sum_{i=0}^{p-1}I(\varsigma_i,\vartheta_i).$$

Set $J_{ij}:=I(\varsigma_i,\vartheta_j)$ so that

$$I=\sum_{i=0}^{p-1}J_{ii}.$$

Note that

$$\epsilon_p:=\eta_{\mathfrak{p}}(-1)=\eta_{\mathfrak{p}}(\xi/\bar{\xi})=\eta_{\mathfrak{p}}(\eta_{\mathfrak{p}}^{\rho})^{-1}(\xi)=\mu_p(\xi)p^{-1}.$$

Then

$$\omega_{1,p}(-1)=\omega_{2,p}(-1)=\epsilon_p^{\delta}$$

where $\delta=0$ or 1 according as j_1-j_2 is even or odd. One sees then that

$$\epsilon_p^{\delta}J_{ij}=J_{(-i)j}=J_{i(-j)}.$$

Since F is right-invariant by $\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}_p$, we see that $J_{ij}=0$ unless $i\equiv\pm j \mod p$. Set

$$\varsigma:=\mathbf{I}_{\zeta^{-1}\mathcal{O}_K}=\sum_{i=0}^{p-1}\varsigma_i\qquad \vartheta:=\mathbf{I}_{\zeta^{-1}\mathcal{O}_K\mathsf{j}}=\sum_{j=0}^{p-1}\vartheta_j.$$

Then

$$I(\varsigma,\vartheta)=J_{00}+\sum_{i=1}^{p-1}(J_{ii}+J_{i(-i)}).$$

Now consider $I(\varsigma_i,\vartheta)$. Since F is right invariant by $\begin{pmatrix}0&1\\-1&0\end{pmatrix}_p$, we have

$$I(\varsigma_i,\vartheta)=I(\hat{\varsigma}_i,\hat{\vartheta}).$$

But $\hat{\vartheta}=p^{1/2}\vartheta_0$ and

$$\hat{\varsigma}_i(x+y\zeta)=p^{-1/2}\psi(-2yi)\mathbf{I}_{\mathbf{Z}_p}(x)\mathbf{I}_{\frac{1}{p}\mathbf{Z}_p}(y)=p^{-1/2}\sum_{u=0}^{p-1}\psi(-2ui/\pi)\varsigma_u.$$

Hence for $i\neq 0$,

$$(1+\epsilon_p^\delta)J_{ii}=J_{ii}+J_{i(-i)}=I(\varsigma_i,\vartheta)=\sum_{u=0}^{p-1}\psi(-2ui/\pi)I(\varsigma_u,\vartheta_0)=J_{00},$$

so that $J_{ii}=J_{00}$ for such i. Finally we see that

$$I=J_{00}+\frac{1}{2}(p-1)J_{00}=\frac{1}{2}(p+1)J_{00}.$$

Consequently, we get:

Theorem B.11. *Suppose* $j_1\equiv j_2 \mod 2$. *Then*

$$\overline{a_{j_1}}a_{j_2}=\frac{1}{C_2}\prod_{q|D_K}\frac{q+1}{2}\cdot\langle\theta_\varphi(\chi_1),\delta^{j_2}f\cdot\theta_{\varphi'}(\chi_2)\rangle$$

where $\theta_\varphi(\chi_1)$ *is the theta lift of the character* χ_1 *from the quadratic space* K *with Schwartz function* $\varphi=\otimes_v\varphi_v$, *where*

$$\varphi_q=\mathbf{I}_{\mathcal{O}_K\otimes\mathbf{Z}_q}\qquad\varphi_\infty(x)=\bar{x}^{m_1}e^{-2\pi|\langle x,x\rangle|}$$

and $\theta_{\varphi'}(\chi_2)$ *is the (holomorphic) theta lift of* χ_2 *from the quadratic space* $(K\mathbf{j},-\langle\cdot,\cdot\rangle)$ *with Schwartz function* $\varphi'=\otimes_v\varphi'_v$ *where*

$$\varphi'_q=\mathbf{I}_{M\otimes\mathbf{Z}_q}\qquad\varphi'_\infty(x\mathbf{j})=\bar{x}^{m_2}e^{-2\pi|\langle x\mathbf{j},x\mathbf{j}\rangle|}$$

with

$$M:=K\mathbf{j}\cap\mathcal{O}_{N^-,N^+}.$$

C. Computing on Shimura curves II: Implementing the Shimizu lifting (by PAUL NELSON)

We consider the problem of computing the values $f(z)$ taken by modular forms f on Shimura curves. To be precise, let f be a eigencuspform for the full Hecke algebra $\mathcal{H}$ on a Shimura curve X, and suppose that one knows the character λ_f of $\mathcal{H}$ associated to f up to some fixed but large precision. If X is non-compact, then f admits a Fourier expansion whose coefficients may be read off from λ_f, but if X is compact, then such Fourier expansions are not available; our aim in this note is to explain in some detail how one can nevertheless compute the values $f(z)$ using the adjoint (as pinned down by Watson) of Shimizu's realization of the Jacquet-Langlands correspondence.

Before doing so, let us review briefly the basic objects of study. Let $\mathbb{H} = \{x + iy : y > 0\}$ be the upper half-plane with the usual action of $\mathbf{GL}_2(\mathbb{R})^+$ (the group of real matrices with positive determinant) by fractional linear transformations. Recall the weight k slash operator on functions $f : \mathbb{H} \to \mathbb{C}$: for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{GL}_2(\mathbb{R})^+$, $f|_k\alpha$ is the function

$$f|_k\alpha(z) = \det(\alpha)^{k/2}(cz + d)^{-k} f(\alpha z). \tag{18}$$

Let k be a positive even integer. Recall that a holomorphic modular form of weight k for a lattice $\Gamma < \mathbf{SL}_2(\mathbb{R})$ is a holomorphic solution $f : \mathbb{H} \to \mathbb{C}$ to the system of functional equations $f|_k\gamma = f$ $(\gamma \in \Gamma)$ that is regular at the cusps of Γ (see [31]). Let $M_k(\Gamma)$ denote the space of such modular forms and $S_k(\Gamma)$ the subspace of cusp forms. Let $\widetilde{\Gamma}$ denote the monoid of all $\alpha \in \mathbf{GL}_2(\mathbb{R})^+$ for which $\Gamma\alpha\Gamma$ is a finite union of either left or right Γ-cosets. Both $M_k(\Gamma)$ and $S_k(\Gamma)$ are modules for the Hecke algebra $\mathcal{H}(\Gamma) = \mathbb{C}[\Gamma\backslash\widetilde{\Gamma}/\Gamma]$ where the action linearly extends $f|_k\Gamma\alpha\Gamma = \sum f|_k\alpha_j$ if $\Gamma\alpha\Gamma = \sqcup\Gamma\alpha_j$. An eigencuspform f, that is to say an eigenfunction of $\mathcal{H}(\Gamma)$ in $S_k(\Gamma)$, corresponds to a character (one-dimensional representation) λ_f of $\mathcal{H}(\Gamma)$ occuring in the module $S_k(\Gamma)$ determined by $f|\varphi = \lambda_f(\varphi)f$ for all $\varphi \in \mathcal{H}(\Gamma)$.

Suppose that one is given $S_k(\Gamma)$ as an abstract $\mathcal{H}(\Gamma)$-module; this situation is a practical one to consider because certain algorithms for computing the spaces $S_k(\Gamma)$ (modular symbols, ...) return it essentially in this form. Then, how can one go about numerically computing the values $f(z)$ $(z \in \mathbb{H})$ for eigencuspforms $f \in S_k(\Gamma)$? To make this question meaningful, we should restrict to Γ for which any eigencuspform f is determined by the associated character λ_f of $\mathcal{H}(\Gamma)$, i.e., for which each character of $\mathcal{H}(\Gamma)$ occurs in $S_k(\Gamma)$ with multiplicity at most one. This condition generally fails if $\widetilde{\Gamma}$ is too small, but is satisfied if Γ is the group of norm one units in an Eichler order in an indefinite rational quaternion algebra B equipped with a fixed real embedding, in which case $\widetilde{\Gamma} = B^\times(\mathbb{Q})^+$ is large and $\Gamma\backslash\mathbb{H}$ is called a Shimura curve. For example, take $B = M_2(\mathbb{Q})$ to be the split (indefinite) rational quaternion algebra. Any Eichler order of level N in $M_2(\mathbb{Q})$ is conjugate to $\left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} \end{smallmatrix}\right)$, whose group of norm one units is denoted $\Gamma_0(N)$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(N)$, any modular form $f \in M_k(\Gamma_0(N))$ admits a Fourier expansion

$$f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z}. \tag{19}$$

If moreover f is an eigencuspform, then one can read off the coefficients $a_f(n)$ (and hence the values $f(z)$) from knowledge of the associated character λ_f of $\mathcal{H}(\Gamma_0(N))$; for example, if n is squarefree and prime to N, then

$$a_f(n) = a_f(1)n^{k/2-1}\lambda_f\left(\Gamma_0(N)\begin{bmatrix} n & \\ & 1\end{bmatrix}\Gamma_0(N)\right). \tag{20}$$

On the other hand, suppose B is a non-split ($B \ncong M_2(\mathbb{Q})$) indefinite ($B \hookrightarrow M_2(\mathbb{R})$) rational quaternion algebra and Γ is the group of norm one units in an Eichler order in B; in that case Γ contains no unipotent elements. Consequently $\Gamma\backslash\mathbb{H}$ has no cusps, $\Gamma\backslash\mathbb{H}$ is a compact Shimura curve, and a Fourier expansion of the shape (19) is not available for forms $f \in M_k(\Gamma)$; thus it is not immediately obvious how to recover the values $f(z)$ of an eigencuspform from its character λ_f.

One way to get around this is via an explicit form of the Jacquet-Langlands correspondence, which specifies a connection between the sets of eigencuspforms in $S_k(\Gamma)$ for Γ arising from the unit groups of orders in *different* quaternion algebras B. First, recall that the set of isomorphism classes of indefinite rational quaternion algebras B is put in bijection with the set of finite even subsets Σ_B of the primes by taking for Σ_B the set of all primes p for which B does not split over $\mathbb{Q}_p$, i.e., for which $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \ncong M_2(\mathbb{Q}_p)$; for details on this and what follows, see [36]. The integer $d_B = \prod_{p\in\Sigma_B} p$ is the reduced discriminant of B. Our assumption that B is indefinite means $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$, so we may fix an embedding $B \hookrightarrow M_2(\mathbb{R})$. Let N be a positive integer prime to d_B. An Eichler order $R_0(N)$ in B of level N is an intersection of two maximal orders in B having the property that for each prime $p \nmid d_B$, there exists an isomorphism $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$ taking $R_0(N) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to the $\mathbb{Z}_p$-order

$$\begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Fix once and for all a representative B, together with a fixed real embedding $B \hookrightarrow M_2(\mathbb{R})$, for each isomorphism class of indefinite rational quaternion algebras. For each such B and each integer N prime to d_B, fix an Eichler order $R_0^{d_B}(N)$ in B and let $\Gamma_0^{d_B}(N)$ denote the group of norm one units in $R_0^{d_B}(N)$; we may and shall take $M_2(\mathbb{Q}) \hookrightarrow M_2(\mathbb{R})$ to be the standard embedding, $R_0^1(N) = R_0(N) := \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z}\end{smallmatrix}\right)$ and $\Gamma_0^1(N) = \Gamma_0(N)$.

Now set $\Gamma' = \Gamma_0^{d_B}(N)$ and $\Gamma = \Gamma_0(d_B N)$. The Jacquet-Langlands correspondence asserts, among other things, that there exists an injection $\mathbb{C}f_B \mapsto \mathbb{C}f$ from the set of eigenspaces for $\mathcal{H}(\Gamma') \circlearrowright S_k(\Gamma')$ to those for $\mathcal{H}(\Gamma) \circlearrowright S_k(\Gamma)$, characterized for instance by a compatibility between the actions of the isomorphic copies in $\mathcal{H}(\Gamma')$ and $\mathcal{H}(\Gamma)$ of the local Hecke algebra $\mathbb{C}[\mathbf{GL}_2(\mathbb{Z}_p)\backslash\mathbf{GL}_2(\mathbb{Q}_p)/\mathbf{GL}_2(\mathbb{Z}_p)]$ for each $p \nmid d_B N$. Shimizu gave an explicit realization of the map "$f_B \otimes f_B \to f$" as a theta correspondence, and Watson pinned down the precise constant of proportionality in the dual correspondence "$f \mapsto f_B \otimes f_B$." An imprecise form of his result asserts that for any $z_1, z_2 \in \mathbb{H}$, one can find a certain (non-holomorphic) theta series Θ_{z_1,z_2} on $\Gamma_0(d_B N)$ such that if $a_f(1) = 1$ and $\langle f, f\rangle = \langle f_B, f_B\rangle$ (with inner products as in the statement of [38, Theorem 1]), so that f is uniquely determined and f_B is normalized up to a scalar of modulus one, then

$$\operatorname{Im}(z_1)^{k/2}\operatorname{Im}(z_2)^{k/2}\overline{f_B(z_1)}f_B(z_2) = \frac{1}{2}\int_{\Gamma_0(d_BN)\backslash\mathbb{H}} \overline{f(z)}\Theta_{z_1,z_2}(z)\, y^{k+1}\frac{dx\,dy}{y^2}. \tag{21}$$

The RHS of (21) is readily calculable. Indeed, let

$$\mathcal{F} = \{x+iy : |x| \leq 1/2, |y| \geq 1\} \tag{22}$$

be the usual fundamental domain for $\Gamma_0(1)\backslash\mathbb{H}$. Substituting the Fourier expansions for $\bar{f}$ and Θ_{z_1,z_2}, we see that their product may be written as an infinite linear combination of terms $e^{2\pi ibx}e^{-2\pi cy}$ for reals b and c. Thus if $d_B = N = 1$, then RHS of (21) is an infinite linear combination of the integrals

$$\int_{\mathcal{F}} y^{k-1}e^{2\pi ibx}e^{-2\pi cy}\,dx\,dy. \tag{23}$$

These integrals decrease rapidly with c, so only a few terms are required to compute (21) to high precision. For general d_B and N, one takes as a fundamental domain for $\Gamma_0(d_BN)\backslash\mathbb{H}$ the essentially-disjoint union $\cup_\gamma\gamma\mathcal{F}$ taken over coset representatives $\gamma \in \Gamma_0(1)/\Gamma_0(d_BN)$; the problem then reduces to computing the Fourier expansions of each $\Gamma_0(1)$-translate of $\bar{f}$ and Θ_{z_1,z_2}. Note that it is *essential* to partition the fundamental domain for $\Gamma_0(d_BN)\backslash\mathbb{H}$ as above because the Fourier expansion at a given cusp converges very slowly away from that cusp (e.g., the series (19) converges slowly if y is too small). In summary, to compute the values of f_B using (21), we must:

(1) Write down the Fourier expansion of f at each cusp of $\Gamma_0(d_BN)$.
(2) Define the theta series Θ_{z_1,z_2}; this is just a matter of extracting its definition from [38, §2].
(3) Write down the Fourier expansion of Θ_{z_1,z_2} at each cusp of $\Gamma_0(d_BN)$.

It is then straightforward to express the RHS of (21) in terms of the integrals (23), which we can evaluate using a general-purpose integrator. The first step is particularly easy when N (and hence d_BN) is squarefree, because then the normalizer in $\Gamma_0(1)$ of $\Gamma_0(d_BN)$, whose action on $S_k(\Gamma_0(d_BN))$ is diagonalizable, acts transitively on the set of cusps of $\Gamma_0(d_BN)$, so one can directly read off the Fourier expansion of an eigencuspform at one cusp from that at another. Since Watson also restricts to the case that N is squarefree in his definition of Θ_{z_1,z_2} and his proof of the formula (21), let us assume henceforth that this is the case.

It will be convenient to work adelically in defining Θ_{z_1,z_2} and computing its Fourier expansion at each cusp, so let us recall that to a function (not necessarily holomorphic) $f : \mathbb{H} \to \mathbb{C}$ that satisfies $f|_k\gamma = f$ for all $\gamma \in \Gamma_0^{d_B}(N)$ one can associate a function $F : B^\times(\mathbb{A}) \to \mathbb{C}$ by the formula

$$F(\gamma g_\infty\kappa_0) = (f|_k g_\infty)(i) \quad \text{if } \gamma \in B^\times(\mathbb{Q}), g_\infty \in B^\times(\mathbb{R})^+, \kappa_0 \in \prod_p (R_0(N)\otimes\mathbb{Z}_p)^\times \tag{24}$$

that satisfies $F(\gamma g\kappa_\infty\kappa_0 z) = \chi_k(\kappa_\infty)F(g)$ for all $\gamma \in B^\times(\mathbb{Q})$, κ_0 as above, z in the center of $B^\times(\mathbb{A})$, and κ_∞ in the stabilizer in $B^\times(\mathbb{R})^+$ of i, where under our fixed isomorphism

$B(\mathbb{R}) \cong M_2(\mathbb{R})$ the character χ_k becomes identified with $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \mapsto e^{ik\theta}$; conversely, one can go from F to f by the formula

$$(25) \qquad y^{k/2} f(x+iy) = F|_{B^\times(\mathbb{R})}(n(x)a(y)), \quad n(x) := \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}, a(y) := \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}.$$

One can similarly pass between functions on $\Gamma_0^{d_B}(N)\backslash\mathbb{H}$ and those on $B^1(\mathbb{Q})\backslash B^1(\mathbb{A})$, where B^1 is the subgroup of norm one elements in $B^\times$. Our discussion above applies in particular when $d_B = 1$, in which case $B^\times = \mathbf{GL}_2$ and $B^1 = \mathbf{SL}_2$.

We turn to the task of defining Θ; for this we must first recall the definition of the Weil representation attached to our indefinite quaternion algebra B, equipped with its reduced norm ν and regarded as a quadratic space (B, ν). Let $\mathbf{e} = \otimes \mathbf{e}_v \in \mathrm{Hom}(\mathbb{A}/\mathbb{Q}, S^1)$ be the standard additive character, characterized by requiring $\mathbf{e}_\infty(x) = e^{2\pi i x}$. Let $\mathcal{S}(V_\mathbb{A}) = \otimes \mathcal{S}(V_v)$ denote the space of Schwarz-Bruhat functions φ on $V(\mathbb{A})$; this space is spanned by pure tensors $\varphi = \otimes\varphi_v$. The group $\mathbf{SL}_2(\mathbb{A})$ acts on $S(V_\mathbb{A})$ by the Weil representation ω: if $\varphi = \otimes\varphi_v \in \mathcal{S}(V_\mathbb{A})$ and $g = (g_v) \in \mathbf{SL}_2(\mathbb{A})$, then $\omega(g)\varphi = \otimes\omega_v(g_v)\varphi_v$, where

$$(26) \qquad \omega_v\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\begin{bmatrix} t & \\ & t^{-1} \end{bmatrix}\right)\varphi_v(\alpha) = |t|_v^{4/2}\mathbf{e}_v(x\nu(\alpha))\varphi_v(t\alpha),$$

$$(27) \qquad \omega_v\left(\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)\varphi_v(\alpha) = (-1)^{\mathbf{1}_{v|d_B}}\mathcal{F}\varphi_v(\alpha),$$

and the Fourier transform $\mathcal{F}\varphi_v(\alpha) := \int_{B(\mathbb{Q}_p)} \varphi(\beta)\mathbf{e}_v(\alpha\beta)\, d\beta$ is defined with respect to a measure $d\beta$ for which $\mathcal{F}\mathcal{F}\varphi_v(\alpha) = \varphi_v(-\alpha)$. (Here $\mathbf{1}_{v|d_B}$ is 1 if v divides d_B and 0 otherwise.) The group $B^\times \times B^\times$ acts (by orthogonal similitudes) on B via the formula $(g_1, g_2)\cdot b = g_1 b g_2^{-1}$, which gives a morphism from the subgroup $B \times_\nu B$ of elements (g_1, g_2) with $\nu(g_1) = \nu(g_2)$ into the orthogonal group of (B, ν). For $\varphi \in \mathcal{S}(V_\mathbb{A})$, $(g_1, g_2) \in (B^\times \times_\nu B^\times)(\mathbb{A})$ and $h \in \mathbf{SL}_2(\mathbb{A})$ define the theta kernel

$$(28) \qquad \theta_\varphi(h; g_1, g_2) = \sum_{\alpha \in B} \omega(h)\varphi(g_1^{-1}\alpha g_2).$$

It is easy to see from the definition of ω and the Poisson summation formula that $h \mapsto \theta_\varphi(h; g_1, g_2)$ is left $\mathbf{SL}_2(\mathbb{Q})$-invariant for any fixed g_1, g_2.

We take $\Theta(h; g_1, g_2) = \theta_\varphi(h; g_1, g_2)$ for the following specific choice of $\varphi = \otimes\varphi_v$. To alleviate notation, let us temporarily set $R = R_0^{d_B}(N)$ and $R_p = R_0^{d_B}(N) \otimes_\mathbb{Z} \mathbb{Z}_p \hookrightarrow B(\mathbb{Q}_p)$. For a finite prime p, let

$$(29) \qquad \varphi_p = \mathrm{vol}(R_p^\times)^{-1}\mathbf{1}_{R_p}$$

where $\mathbf{1}_{R_p}$ is the characteristic function of R_p and vol is defined with respect to a measure on $B^\times(\mathbb{Q}_p)$ assigning volume one to the unit group of a maximal order. Since we have assumed that N is squarefree, we have $\varphi_p = \mathbf{1}_{R_p}$ if $p \nmid d_B N$ and $\varphi_p = (p+1)\mathbf{1}_{R_p}$ if $p \nmid d_B$, $p|N$. Let

$$(30) \qquad \varphi_\infty = X^k e^{-2\pi P}$$

with X, P given by

$$X\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}(a+d+i(b-c)), \quad P\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{1}{2}(a^2+b^2+c^2+d^2). \tag{31}$$

under our fixed identification $B(\mathbb{R}) \cong M_2(\mathbb{R})$. We remark that the positive definite quadratic form $P : M_2(\mathbb{R}) \to \mathbb{R}$ is a majorant for the $(2,2)$-signature determinant form; there is a four-dimensional real manifold of such majorants and a more intrinsic definition of this particular choice of X and P goes as follows. Let $\iota : \mathbb{C} \to M_2(\mathbb{R})$ be the $\mathbb{R}$-algebra embedding that sends i to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let $\varepsilon = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, which has the property that $\varepsilon^2 = 1$ and $\varepsilon\iota(z)\varepsilon = \iota(\bar{z})$ for each $z \in \mathbb{C}$. Define $X, Y : M_2(\mathbb{R}) \to \mathbb{C}$ by requiring that $\alpha = \iota(X(\alpha)) + \varepsilon\iota(Y(\alpha))$ for each $\alpha \in M_2(\mathbb{R})$. Then $\det = |X|^2 - |Y|^2$ and $P = |X|^2 + |Y|^2$. In other words, P is the unique positive-definite majorant for $M_2(\mathbb{R})$ regarded as a $(1,1)$-signature hermitian space over $\mathbb{C}$ via the embedding ι.

We are now prepared to state a precise form of the identity (21). Let N be a squarefree integer prime to d_B, let $\Gamma = \Gamma_0(d_B N)$, let $\Gamma' = \Gamma_0^{d_B}(N)$, and let $f \in S_k(\Gamma)$ and $f_B \in S_k(\Gamma')$ be eigencuspforms such that f is the Jacquet-Langlands lift of f_B. Define $F : \mathbf{GL}_2(\mathbb{A}) \to \mathbb{C}$ and $F_B : B^\times(\mathbb{A}) \to \mathbb{C}$ by the recipe (24). We normalize so that $a_f(1) = 1$ and $\langle F_B, F_B \rangle = \langle F, F \rangle$ with the inner products as in the statement of [38, Theorem 1]. Choose a Haar measure dh on $\mathbf{SL}_2(\mathbb{Q})\backslash\mathbf{SL}_2(\mathbb{A})$ so that if $G : \mathbf{SL}_2(\mathbb{Q})\backslash\mathbf{SL}_2(\mathbb{A}) \to \mathbb{C}$ is right-invariant under $\mathbf{SO}(2) \times \prod_p \mathbf{SL}_2(\mathbb{Z}_p)$, then

$$\int_{\mathbf{SL}_2(\mathbb{Q})\backslash\mathbf{SL}_2(\mathbb{A})} G(h)\, dh = \int_{\mathcal{F}} G|_{\mathbf{SL}_2(\mathbb{R})}(n(x)a(y))\, \frac{dx\, dy}{y^2} \tag{32}$$

with $\mathcal{F}$ the fundamental domain (22) for $\mathbf{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and $n(x), a(y)$ as in (25). Then [38, Theorem 1] implies that for any $g_1, g_2 \in (B^\times \times_\nu B^\times)(\mathbb{A})$, we have

$$\overline{F_B(g_1)} F_B(g_2) = \frac{1}{2} \int_{\mathbf{SL}_2(\mathbb{Q})\backslash\mathbf{SL}_2(\mathbb{A})} \overline{F(h)}\Theta(h; g_1, g_2)\, dh. \tag{33}$$

This formula applies in particular when $g_1, g_2 \in B^1(\mathbb{A})$ have norm one, which is the only case that we shall need to consider. The integrand on the RHS of (33) is right-invariant under the subgroup

$$K_0(d_B N) := \mathbf{SO}(2) \times \prod_p (R_0(d_B N) \otimes \mathbb{Z}_p)^1 \tag{34}$$

of $\mathbf{SL}_2(\mathbb{A})$, so the integral descends to $\Gamma_0(d_B N)\backslash\mathbb{H}$ and we see that (21) holds with

$$\Theta_{z_1,z_2}(z) = c \sum_{\alpha \in R_0^{d_B}(N)} X(\alpha')^k e^{2\pi i x \nu(\alpha')} e^{-2\pi y P(\alpha')}, \quad \alpha' := g_1^{-1}\alpha g_2 \tag{35}$$

where $g_1, g_2 \in B(\mathbb{R})$ are chosen so that $g_j \cdot i = z_j$ $(j = 1, 2)$ and $c = [\Gamma_0(N) : \Gamma_0(d_B N)]^{-1}$. This is a theta series attached to the rational quaternary quadratic form $R_0^{d_B}(N) \ni \alpha \mapsto \nu(\alpha')$ of signature $(2,2)$ and the spherical harmonic $\alpha \mapsto X(\alpha')^k$ of degree k. The inversion formulae for such series, which allow one to compute their Fourier expansions at various cusps, are conveniently packaged into the machinery of the Weil representation. To put this into practice, let us take as a fundamental domain for $\mathbf{SL}_2(\mathbb{Q})\backslash\mathbf{SL}_2(\mathbb{A})$ the essentially-disjoint union of the sets $\mathcal{G}_d$, indexed by the divisors d of the squarefree integer $d_B N$, defined as follows. Let w denote the Weyl element $w = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ of $\mathbf{SL}_2$. Denote temporarily by U_p the subgroup $(R_0(d_B N) \otimes \mathbb{Z}_p)^1$ of $\mathbf{SL}_2(\mathbb{Z}_p)$. Let $\mathcal{G}_d$ be the set of all $g = (g_v) \in \mathbf{SL}_2(\mathbb{A})$ for which

- $g_\infty = n(x)a(y)\kappa_\infty$ with $x + iy$ in the standard fundamental domain $\mathcal{F}$ for $\mathbf{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and $\kappa_\infty \in \mathbf{SO}(2)$,
- $g_p \in U_p$ if $p \nmid d_B N$, and
- $g_p \in n(a)wU_p$ for some $a \in \{0, 1, \dots, p-1\}$ if $p \nmid d_B N$.

Thus we have partitioned (g_v) according to whether, for each prime divisor p of $d_B N$, the reduction of g_p modulo p belongs to the big or little Bruhat cell of $\mathbf{SL}_2(\mathbb{Z}/p)$. Let $w(d) \in \prod_p \mathbf{SL}_2(\mathbb{Z}_p)$ be such that $w(d)_p = 1$ if $p \nmid d$ and $w(d)_p = w$ if $p|d$. Define

$$G_d(x+iy) = \overline{F(n(x)a(y) \times w(d))}\Theta(n(x)a(y) \times w(d); g_1, g_2). \tag{36}$$

Then the integral on the RHS of (33) is just

$$\frac{1}{[\Gamma_0(1) : \Gamma_0(d_B N)]} \sum_{d|d_B N} \int_{\mathcal{F}} \sum_{a \in \mathbb{Z}/d\mathbb{Z}} G_d(z+a)\, \tfrac{dx\, dy}{y^2},$$

so the problem is essentially to compute the usual Fourier expansion of G_d at ∞. To do so, we shall compute separately $F(n(x)a(y) \times w(d))$ and $\Theta(n(x)a(y) \times w(d))$.

Let us recall the formula for $F(n(x)a(y) \times w(d))$ given by Atkin-Lehner theory. Let

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \tag{37}$$

be the usual Fourier expansion for f, which we have normalized so that $\lambda_f(1) = 1$ and the Deligne bound reads $|\lambda_f(p)| \leq 2$. For a prime $p|d_B N$ we have $\varepsilon_f(p) := p^{1/2}\lambda_f(p) \in \{\pm 1\}$. For any divisor $d|d_B N$, let $\varepsilon_f(d) := \prod_{p|d} \varepsilon_f(p)$; then

$$F(n(x)a(y) \times w(d)) = \mu(d)\varepsilon_f(d) \left(\frac{y}{d}\right)^{k/2} f\left(\frac{z}{d}\right) = \frac{\mu(d)}{a_f(d)} y^{k/2} \sum_{n \geq 1} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i \frac{n}{d} z}. \tag{38}$$

We turn now to $\Theta(n(x)a(y) \times w(d))$; see [38, §2] for details. Since in general

$$\Theta(hh'; g_1, g_2) = \theta_\varphi(hh'; g_1, g_2) = \theta_{\omega(h')\varphi}(h; g_1, g_2), \tag{39}$$

let us first compute $\omega(w(d))\varphi$. For $v \nmid d$ we have $\omega_v(w(d)_v)\varphi_v = \varphi_v$. If $v = p|d$, then $w(d)_p = w$, so $\omega_p(w(d)_p)\varphi_p = (-1)^{\mathbf{1}|_{p|d_B}} \mathcal{F}\varphi_p$; we have

$$\mathcal{F}\mathbf{1}_{R_0^{d_B}(N) \otimes \mathbb{Z}_p} = p^{-1}\mathbf{1}_{R_0^{d_B}(N;d) \otimes \mathbb{Z}_p}$$

for some lattice that we denote by $R_0^{d_B}(N; d)$ for lack of imagination: if $p|\gcd(d, N)$, the isomorphism $B(\mathbb{Q}_p) \cong M_2(\mathbb{Q}_p)$ taking $R_0^{d_B}(N) \otimes \mathbb{Z}_p$ to $\left(\begin{smallmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right)$ takes $R_0^{d_B}(N; d)$ to $\left(\begin{smallmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{smallmatrix}\right)$, while if $p|\gcd(d,d_B)$ then $R_0^{d_B}(N;d)\otimes\mathbb{Z}_p$ is the inverse of the unique maximal ideal in $B(\mathbb{Q}_p)$. Let $\psi(N)=\prod_{p|N}(p+1)=[\Gamma_0^{d_B}(1):\Gamma_0^{d_B}(N)]$. Then

$$\Theta(n(x)a(y)\times w(d);g_1,g_2)=\frac{\mu((d,d_B))\psi(N)}{d}y^{k/2+1}\sum_{\alpha\in R_0^{d_B}(N;d)}[X^k e^{2\pi i x\nu}e^{-2\pi yP}](\alpha') \tag{40}$$

with $\alpha'=g_1^{-1}\alpha g_2$ as before, and so

$$\begin{aligned}\frac{1}{[\Gamma_0(1):\Gamma_0(d_BN)]}\sum_{a\in\mathbb{Z}/d\mathbb{Z}}G_d(z+a)&=\frac{y^{k+1}\mu((d,N))}{\psi(d_B)a_f(d)}\\&\quad\times\sum_{n\in\mathbb{N}}\sum_{\substack{\alpha\in R_0^{d_B}(N;d)\\ n/d-\nu(\alpha')\in\mathbb{Z}}}a_f(n)e^{2\pi i\frac{n}{d}(-x+iy)}[X^k e^{2\pi i x\nu}e^{-2\pi yP}](\alpha').\end{aligned} \tag{41}$$

Summing over $d|d_BN$ and integrating over $\mathcal{F}$ gives, finally, the formula

$$\overline{F_B(g_1)}F_B(g_2)=\frac{1}{2}\sum_{d|d_BN}\frac{\mu((d,N))}{\psi(d_B)a_f(d)}\sum_{\substack{n\in\mathbb{N}\\ \alpha\in R_0^{d_B}(N;d)\\ n/d-\nu(\alpha')\in\mathbb{Z}}}\sum a_f(n)X^k(\alpha')\mathcal{I}_{k-1}\left[\nu(\alpha')-\frac{n}{d},P(\alpha')+\frac{n}{d}\right], \tag{42}$$

where $\mathcal{I}_a(b,c):=\int_{\mathcal{F}}y^a e^{2\pi i bx}e^{-2\pi cy}\,dx\,dy$.

The computational challenges involved in making this formula practical are

(1) computing the coefficients $a_f(n)$ (which can be done by several software packages, such as SAGE)
(2) determining an explicit basis for the lattice $R_0^{d_B}(N;d)$,
(3) rapidly enumerating those vectors $\alpha\in R_0^{d_B}(N;d)$ for which $P(\alpha')$ is small, and
(4) rapidly computing the integrals $\mathcal{I}_a(b,c)$, or caching them for later reuse.

Some further topics worth addressing include

(1) computing derivatives of F_B by differentiating both sides of (28) with respect to g_1 or g_2, which amounts to replacing ϕ_∞ by its image under a suitable raising operator (I've already implemented this),
(2) factorizing the rank four lattice $g_1^{-1}R_0^{d_B}(N;d)g_2$ into "sums of products of" rank two lattices for special values of g_1,g_2 (CM points) and using this to automate the derivation of explicit relations between values/periods of modular forms on Shimura curves and special values of L-series (à la Waldspurger),
(3) spelling out all of the above over a totally real field,
(4) ...?

I have written some SAGE code to implement the formula (42) and its generalization in which one evaluates the ℓ_1th derivative with respect to z_1 and ℓ_2th derivative with respect to z_2 of $\overline{f_B(z_1)}f_B(z_2)$. Here are some example illustrating a few simple cases. (Because the software is still under development, the notation might be a bit inconsistent between the various examples.)

Example C.1. Take $d_B = N = 1$, $k = 12$, and

$$\Delta = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n \in S_{12}(1), \quad q := e^{2\pi i z}$$

the Ramanujan function. One obtains $\Delta(i) \approx 0.00178536985064215$ by summing the first 9 terms in its series expansion and gets the same answer after summing 20 terms. Thus $|\Delta(i)|^2 \approx 3.18754550358197e{-6}$. The formula above gives $|\Delta(i)|^2 = \frac{1}{2} \int \overline{\Delta} \Theta$ for a theta series Θ. Here is some (hopefully self-explanatory) output from a session with SAGE illustrating this, using some code I've written; we get the answer correct to one part in a billion. (The unused parameters l_1, l_2 can be used to take derivatives.)

```
sage: delta = Form(CuspForms(1,12).0,20); delta
Form of level 1 and weight 12 and precision 20.
sage: delta.evaluate(I)^2
3.18754550358197e-6
sage: delta.at_prec(9).evaluate(I)^2
3.18754550358197e-6
sage: delta.dual()
Form of level 1 and weight -12 and precision 20.
sage: theta = Theta(OO(1),12,0,0,I,I,11); theta
Theta series with k=12, z1=I, z2=I, l1=0, l2=0, prec=11
associated to an order of discriminant 1 with basis:
[1 0]
[0 0]
[0 1]
[0 0]
[0 0]
[1 0]
[0 0]
[0 1]
sage: delta.dual()*theta
Product of:
 * Form of level 1 and weight -12 and precision 20.
 * Theta series with k=12, z1=I, z2=I, l1=0, l2=0, prec=11
   associated to an order of discriminant 1 with basis:
[1 0]
[0 0]
[0 1]
```

```
[0 0]
[0 0]
[1 0]
[0 0]
[0 1]
sage: 0.5*(delta.dual()*theta).integrate()
3.18754550643e-06 - 8.70739974961e-22*I
sage: (0.5*(delta.dual()*theta).integrate()) / delta.evaluate(I)^2
1.00000000089 - 2.73169425811e-16*I
```

Example C.2. Let's give an example that involves computing on a non-split quaternion algebra. (A small part of my notes on this example, such as the generating set for the unit group of the maximal order, may have been taken from some paper; if so, I can't remember which one.) Define elements $i, j, k \in M_2(\mathbb{R})$:

$$i = \begin{bmatrix} \sqrt{2} & \\ & -\sqrt{2} \end{bmatrix}, \quad j = \begin{bmatrix} & 1 \\ -3 & \end{bmatrix}, \quad k = ij = \begin{bmatrix} 0 & \sqrt{2} \\ 3\sqrt{2} & 0 \end{bmatrix}. \tag{43}$$

Then, identifying $\mathbb{Q}$ with the subalgebra of diagonal matrices in $M_2(\mathbb{R})$ we have $i^2 = 2$, $j^2 = -3$, and $ij = -ji$. The rational quaternion algebra $B := \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$ is the unique such algebra of reduced discriminant $d_B = 6$. If we set

$$e_1 = 1 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad e_2 = \frac{i+k}{2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{2} & -1/\sqrt{2} \end{bmatrix},$$

$$e_3 = \frac{1+j}{2} = \begin{bmatrix} 1/2 & 1/2 \\ -3/2 & 1/2 \end{bmatrix}, \quad e_4 = k = \begin{bmatrix} 0 & \sqrt{2} \\ 3\sqrt{2} & 0 \end{bmatrix},$$

then $R_0^6(1) := \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$ is a maximal order in B (i.e., an Eichler order of level 1). The group $\Gamma_0^6(1)$ of norm one units is generated by

$$\gamma_1 = \frac{i+2j-k}{2} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2}+1 \\ -3/\sqrt{2}-3 & -1/\sqrt{2} \end{bmatrix}, \quad \gamma_2 = \frac{i-2j+k}{2} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2}-1 \\ 3/\sqrt{2}+3 & -1/\sqrt{2} \end{bmatrix},$$

$$\gamma_3 = \frac{1+j}{2} = \begin{bmatrix} 1/2 & -1/2 \\ -3/2 & 1/2 \end{bmatrix}, \quad \gamma_4 = \frac{1+3j-2k}{2} = \begin{bmatrix} 1/2 & -\sqrt{2}+3/2 \\ -3\sqrt{2}-9/2 & 1/2 \end{bmatrix},$$

The space $S_4(\Gamma_0(6))$ is one-dimensional. Let $f = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + \cdots$ be its unique normalized eigencuspform. By the (inverse?) Jacquet-Langlands correspondence, f transfers to an eigencuspform $f_B \in S_4(\Gamma_0^6(1))$, which we may normalize as above up to a scalar of modulus one. We shall compute its value at a CM point and check its rationality up to a certain transcendental factor. The element

$$\alpha_4 := \gamma_1^2\gamma_2 = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2}+1 \\ -3/\sqrt{2}-3 & 1/\sqrt{2} \end{bmatrix} \tag{44}$$

satisfies $\alpha_4^2 + 1 = 0$, and so induces an optimal embedding $\mathbb{Z}[i] \hookrightarrow R_0^6(1)$. Its unique fixed point in $\mathbb{H}$ under our fixed real embedding of B is

$$\tau_4 := \frac{\sqrt{2}-1}{3}(1+i\sqrt{2}) \approx 0.138071 + 0.195262i. \tag{45}$$

Define $\Omega_4 = 2.62205755429212$; this is some period for an elliptic curve with CM by $\mathbb{Q}(i)$. In the SAGE session quoted below, we compute (using the integral formula that expresses $|f_B(\tau_4)|^2$ as the Petersson inner product of f with a certain theta series) that

$$\left(\frac{\pi}{\Omega_4}\right)^{2((4+1)-1)} |f_B(\tau_4)|^2 \approx 0.000289351851518. \tag{46}$$

Note that

$$1/0.000289351851518 \approx 3456.00000398750, \tag{47}$$

which is suspiciously close to the integer $3456 = 2^7 \cdot 3^3$.

```
sage: f = Form(CuspForms(6,4).0,5); f
Form of level 6 and weight 4 and precision 5.
sage: tau4
1/3*(I*sqrt(2) + 1)*(sqrt(2) - 1)
sage: alpha4
[    -1/2*rt2 -1/2*rt2 + 1]
[-3/2*rt2 - 3      1/2*rt2]
sage: (act(alpha4,tau4) - tau4).n()
8.32667268468867e-17 + 1.66533453693773e-16*I
sage: alpha4^2
[-1  0]
[ 0 -1]
sage: omega4
2.62205755429212
sage: theta = Theta(OO(6),4,0,0,tau4,tau4,9); theta
Theta series with k=4, z1=1/3*(I*sqrt(2) + 1)*(sqrt(2) - 1),
z2=1/3*(I*sqrt(2) + 1)*(sqrt(2) - 1), l1=0, l2=0, prec=9
associated to an order of discriminant 6 with basis:
[1 0]
[0 1]
[ 1/2*rt2  1/2*rt2]
[ 3/2*rt2 -1/2*rt2]
[ 1/2  1/2]
[-3/2  1/2]
[    0   rt2]
[3*rt2     0]
```

```
sage: f.dual()*theta
Product of:
 * Form of level 6 and weight -4 and precision 5.
 * Theta series with k=4, z1=1/3*(I*sqrt(2) + 1)*(sqrt(2) - 1),
   z2=1/3*(I*sqrt(2) + 1)*(sqrt(2) - 1), l1=0, l2=0, prec=9
   associated to an order of discriminant 6 with basis:
[1 0]
[0 1]
[ 1/2*rt2  1/2*rt2]
[ 3/2*rt2 -1/2*rt2]
[ 1/2  1/2]
[-3/2  1/2]
[    0   rt2]
[3*rt2     0]
sage: (pi.n()/omega4)^(2*((4+1)-1))*0.5*(f.dual()*theta).integrate()
0.000289351851518 - 1.24523937669e-20*I
sage: 1/0.000289351851518
3456.00000398750
sage: 3456.factor()
2^7 * 3^3
```

Example C.3. Here's a quick example illustrating the rationality of derivatives for $f \in S_2(11)$. We first evaluate directly via the q-expansion and then via the Shimizu correspondence. The functionality illustrated below also works on non-split quaternion algebras; we do it first on $\mathbf{GL}_2$ to demonstrate correctness.

```
sage: f = Form(CuspForms(11,2).0,prec); f
Form of level 11 and weight 2 and precision 200:
q - 2*q^2 - q^3 + 2*q^4 + q^5 + O(q^6)
sage: tau = EllipticCurve('11a').heegner_point(-7,1).tau(); tau; tau=tau.n()
1/22*sqrt_minus_7 - 9/22
sage: omega = EllipticCurve([1,-1,0,-2,-1]).period_lattice().real_period();
omega
1.93331170561681
sage: [imag(tau)^(-4/2)*(2*pi.n()*I/omega)^(2*(2+2*j))/(-4*pi.n()*imag(tau))
^(2*j)  * abs(f.deriv(j).evaluate(tau))^2 for j in [0,1,2]]
[77.000000000000, 847.00000000000, 232925.00000000]
sage: [imag(tau)^(-4/2)*(2*pi.n()*I/omega)^(2*(2+2*j))/(-4*pi.n()*imag(tau))
^(2*j) * 0.5*(f.at_prec(6).dual()*Theta(OO(1,11),11,2,j,j,tau,tau,12)).
integrate(ii) for j in [0,1,2]]
[76.9999999999160 - 5.26694055493770e-11*I, 847.000000228287 -
5.04460998029114e-12*I, 232925.000257534 + 1.95733349388877e-7*I]
```

Example C.4. Let $f \in S_2(15)$ be the elliptic curve of conductor 15. Then f transfers to f_B on the indefinite quaternion algebra B of discriminant 15. We compute a maximal order R in B, an optimal embedding into R of the maximal order in $\mathbb{Q}(\sqrt{-7})$, the fixed point τ of $(1+\sqrt{-7})/2$ under this embedding, and the absolute value squared of f_B at τ; we end up with the rational number $7^2/(2^5.3^2.5)$.

```
sage: O = OO(15,1)
sage: g = O.basis[2] + O.basis[3]
sage: g^2 - g + 2 == 0
True
sage: tau = (-1/5*(-sqrt(-7) + sqrt(3))/(sqrt(3) - 1)) # fixed by g
sage: omega = EllipticCurve([1,-1,0,-2,-1]).period_lattice().real_period();
omega
1.93331170561681
sage: f = Form(CuspForms(15,2).0,5)
sage: theta = Theta(O,1,2,0,0,tau,tau,prec2)
sage: f.dual()*theta
Product of:
 * Form of level 15 and weight -2 and precision 5:
   q - q^2 - q^3 - q^4 + q^5 + O(q^6)
 * Theta series with k=2, z1=1/5*(sqrt(-7) - sqrt(3))/(sqrt(3) - 1),
   z2=1/5*(sqrt(-7) - sqrt(3))/(sqrt(3) - 1), l1=0, l2=0, prec=10
   associated to an order of discriminant 15.1 with basis:
[1 0]
[0 1]
[ rt3    0]
[   0 -rt3]
[1/2 1/2]
[5/2 1/2]
[ 1/2*rt3  1/2*rt3]
[-5/2*rt3 -1/2*rt3]
sage: (2*pi.n()/omega)^(2*2)*0.5*(f.dual()*theta).integrate(ii)
0.0340277777778 + 2.37565437957e-17*I
sage: guess = 7^2 / (2^5 * 3^2 * 5)
sage: guess.n()
0.0340277777777778
```

Example C.5. Finally, let's do an example involving a non-maximal Eichler order in a non-split quaternion algebra. Let $f \in S_2(30)$ be the newform of weight 2 and level 30, and let R be an Eichler order of level 2 in the quaternion algebra B of discriminant 15. Then f transfers to a form f_B invariant under the norm one units in R. We compute $f_B(\tau)$ for τ as in the previous example and get $7^2/(2^6.3.5)$.

```
sage: f = Form(CuspForms(30,2).newforms()[0],5)
sage: O = OO(15,2)
sage: g = O.basis[2] + O.basis[3]
sage: g^2 - g + 2 == 0
True
sage: tau = (-1/5*(-sqrt(-7) + sqrt(3))/(sqrt(3) - 1)).n()
sage: omega = EllipticCurve([1,-1,0,-2,-1]).period_lattice().real_period()
sage: theta = Theta(O,2,2,0,0,tau,tau,10)
sage: f.dual()*theta
Product of:
 * Form of level 30 and weight -2 and precision 5:
   q - q^2 + q^3 + q^4 - q^5 + O(q^6)
 * Theta series with k=2, z1=-0.473205080756888 + 0.722832700602043*I,
   z2=-0.473205080756888 + 0.722832700602043*I, l1=0, l2=0, prec=10
   associated to an order of discriminant 15.2 with basis:
[1 0]
[0 1]
[ rt3    0]
[   0 -rt3]
[ 1/2*rt3 + 1/2  1/2*rt3 + 1/2]
[-5/2*rt3 + 5/2 -1/2*rt3 + 1/2]
[   rt3    rt3]
[-5*rt3   -rt3]
sage: (2*pi.n()/omega)^(2*2)*0.5*(f.dual()*theta).integrate(ii)
0.0510416666666 + 1.51291501324e-13*I
sage: guess = 7^2 / (2^6 * 3 * 5)
sage: guess.n()
0.0510416666666667
```

D. Norms of definite quaternionic modular forms in Magma: A primer (by JOHN VOIGHT)

This document gives a very brief introduction to computing Hilbert modular forms using Brandt matrices in Magma, with an application to computing the norm of a form as it arises in a definite quaternion algebra via the Jacquet-Langlands correspondence.

The algorithm employed is due to Dembélé [5] with Donnelly [6], who also put in substantial work in implementation, and they use algorithms for quaternion algebras implemented by the author [37] with Kirschmer [16] and Donnelly. They are intended for use at the Arizona Winter School 2011 and probably don't explain everything very well—so if you have any questions, please ask!

To load magma where it is installed, type `magma`. Inside the session, you can load this demo by typing `iload magma-defquat-demo.m`, assuming you have the demo file. (You need a sufficiently new version of Magma as well.)

D.1. **A space of Hilbert cusp forms.** We begin by first defining the field $\mathbf{Q}(\sqrt{5})$ and computing its ring of integers $\mathbf{Z}_F$.

```
> _<x> := PolynomialRing(Rationals());
> F<w> := NumberField(x^2-5);
> w^2;
5
> MinimalPolynomial(w);
x^2 - 5;
> Z_F := Integers(F);
```

We are now ready to define a space of cusp forms.

```
> M := HilbertCuspForms(F, 17*Z_F);
> M;
Cuspidal space of Hilbert modular forms over
    Number Field with defining polynomial x^2 - 5 over the Rational Field
    Level: Ideal of norm 289 generated by [17, 0]
    Weight: [ 2, 2 ]
```

Nothing has been computed so far—it has just created the hull of the space $S_2(17)$ of Hilbert cusp forms of parallel weight $(2,2)$ (the default) and level $(17) = 17\mathbf{Z}_F$. Note that 17 is inert in F.

Now we compute the dimension and the action of a Hecke operator.

```
> Dimension(M);
5
> T2 := HeckeOperator(M, 2*Z_F);
> T2;
[-1  0  2  1 -2]
```

```
[-1  2 -1  0  1]
[ 2  0 -1  1 -2]
[ 1  0  1  1 -2]
[-1  1 -1  0  0]
> CharacteristicPolynomial(T2);
x^5 - x^4 - 14*x^3 + 2*x^2 + 25*x + 3
> Factorization($1);
[
    <x + 3, 1>,
    <x^4 - 4*x^3 - 2*x^2 + 8*x + 1, 1>
]
```

The command `$1` takes the output from the previous line. (Your T_2 may vary, but the characteristic polynomial should be the same!)

Therefore, the space $S_2(17)$ breaks up into at least two Hecke irreducible subspaces. We can compute the complete decomposition as follows.

```
> newforms := NewformDecomposition(M);

>> newforms := NewformDecomposition(M);
                                    ^
Runtime error in 'NewformDecomposition': Currently implemented only for new
spaces (constructed using NewSubspace)
```

Oops! We need to work in the new subspace. (This is a design decision: that the old subspaces are not interesting and so should be factored out automatically.)

In this case, the space $S_2(1)$ has dimension zero, so all forms are new, but nevertheless we should compute this directly.

```
> S := NewSubspace(M);
> S;
New cuspidal space of Hilbert modular forms over
    Number Field with defining polynomial x^2 - 5 over the Rational Field
    Level: Ideal of norm 289 generated by [17, 0]
    New level: Ideal of norm 289 generated by [17, 0]
    Weight: [ 2, 2 ]
> Dimension(S);
5
> newforms := NewformDecomposition(S);
> newforms;
[*
   Cuspidal newform space of Hilbert modular forms over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
```

```
      Level: Ideal of norm 289 generated by [17, 0]
      New level: Ideal of norm 289 generated by [17, 0]
      Weight: [ 2, 2 ]
      Dimension 1,
   Cuspidal newform space of Hilbert modular forms over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
      Level: Ideal of norm 289 generated by [17, 0]
      New level: Ideal of norm 289 generated by [17, 0]
      Weight: [ 2, 2 ]
      Dimension 4
*]
```

So there are indeed two Hecke irreducible spaces, one of dimension 1 (corresponding to an elliptic curve over F of conductor (17)) and one of dimension 4.

Let's grab a constituent eigenform from each space and compute their Hecke eigenvalues.

```
> f := Eigenform(newforms[1]);
> g := Eigenform(newforms[2]);
> [[* Norm(pp), HeckeEigenvalue(f,pp) *] : pp in PrimesUpTo(20,F)];
[ [* 4, -3 *], [* 5, -2 *], [* 9, -6 *], [* 11, 0 *], [* 11, 0 *],
[* 19, -4 *], [* 19, -4 *] ]
```

This says for example that $a_{(2)}(f) = -3$, $a_{(\sqrt{5})} = -2$, etc. One can identify this form as the base change from $\mathbf{Q}$ of the unique classical elliptic cusp form of weight 2 and level 17—i.e., it corresponds to $X_0(17)$.

We can similarly compute with form g.

```
> [[* Norm(pp), HeckeEigenvalue(g,pp) *] : pp in PrimesUpTo(20,F)];
[ [* 4,
    e
*], [* 5,
    1/2*(e^3 - 5*e^2 + e + 7)
*], [* 9,
    e^2 - 3*e
*], [* 11,
    1/2*(-e^3 + 4*e^2 - e - 4)
*], [* 11,
    1/2*(-e^3 + 4*e^2 - e - 4)
*], [* 19,
    -e^3 + 3*e^2 + 3*e - 3
*], [* 19,
    -e^3 + 3*e^2 + 3*e - 3
*] ]
```

```
> H<e> := CoefficientField(g);
> H;
Number Field with defining polynomial x^4 - 4*x^3 - 2*x^2 + 8*x + 1 over the
Rational Field
> MinimalPolynomial(e);
x^4 - 4*x^3 - 2*x^2 + 8*x + 1
> Discriminant(H);
99584
> IsTotallyReal(H);
true
```

So we see that the field H generated by the Hecke eigenvalues of g is a totally real quartic field of discriminant 99584.

Exercise D.1. *What is Hecke ring and the Eisenstein ideal associated to the above form g?*

What is the smallest conductor of a modular elliptic curve over $\mathbf{Q}(\sqrt{7})$?

If you know about modularity and base change, prove that f is indeed associated to the base change of $X_0(17)$ to F.

D.2. **Brandt matrices, inner products.** We continue with the example from above. We wish to compute with inner products. We first must define a few functions for this purpose (which will probably make their way into the next release of Magma...). The code itself can be safely ignored on a first reading (but must still be entered).

```
Coordinates := function(f);
  fc := f`coords_raw;
  fc *:= Lcm([Denominator(fci) : fci in Eltseq(fc)]);
  return fc;
end function;

Content := function(fc);
  E := BaseRing(fc);
  Z_E := Integers(E);
  gg := ideal<Z_E | [fci : fci in Eltseq(fc)]>;
  return gg;
end function;

InnerProductVec := function(f,g : Normalized := true);
  fc := Coordinates(f);
  gc := Coordinates(g);
  Ef := BaseRing(fc);
  Eg := BaseRing(gc);
  Efg := CompositeFields(Ef, Eg)[1];
```

```
  Z_Efg := Integers(Efg);
  M := Parent(f);
  while assigned M`Ambient do
    M := M`Ambient;
  end while;
  Mmat := M`InnerProductBig;
  innprod := (ChangeRing(fc, Efg)*
                ChangeRing(Mmat, Efg)*
                Transpose(Matrix(ChangeRing(gc, Efg))))[1];
  if Normalized then
    if Type(Z_Efg) eq RngInt then
      innprod := innprod/(Generator(Content(fc))*Generator(Content(gc)));
      innprod := (Integers()!innprod)*Integers();
    else
      innprod := ((Z_Efg!innprod)*Z_Efg)
                        /(Z_Efg!!Content(fc))/(Z_Efg!!Content(gc));
    end if;
  else
    innprod := innprod*Integers(Efg);
  end if;
  return innprod;
end function;
```

We illustrate the use of these functions.

```
> Coordinates(f);
( 1  0 -1  0  0  0)
> Coordinates(g);
(1 1/2*(-3*e^3 + 6*e^2 + 21*e + 6) 1 -e^3 + 3*e^2 + 5*e - 3
1/2*(e^2 - 4*e - 3)  e^3 - 3*e^2 - 4*e)
```

These are the coordinates for f and g with respect to a choice of basis for the space $M_2(17)$. Note that this space is 6 dimensional, due to the presence of the 1-dimensional space of Eisenstein series.

```
> InnerProductVec(f,f);
Ideal of Integer Ring generated by 12
> InnerProductVec(f,g);
Zero Principal Ideal
Generator:
    [0, 0, 0, 0]
> InnerProductVec(g,g);
Principal Ideal
```

```
Generator:
    [780, 912, 972, -276]
```

We should factor the last inner product.

```
> innfact := Factorization($1);
> innfact;
[
    <Prime Ideal
    Two element generators:
        [2, 0, 0, 0]
        [1, 0, 1, 0], 9>,
    <Prime Ideal
    Two element generators:
        [29, 0, 0, 0]
        [24, 1, 0, 0], 1>,
    <Principal Prime Ideal
    Generator:
        [3, 0, 0, 0], 1>,
    <Prime Ideal
    Two element generators:
        [389, 0, 0, 0]
        [279, 1, 0, 0], 1>
]
> [Norm(pp[1]) : pp in innfact];
[ 2, 29, 81, 389 ]
> for pp in innfact do
>   bl, gen := IsPrincipal(pp[1]);
>   if bl then print Norm(pp[1]), H!gen; end if;
> end for;
2 1/2*(-e^3 + 4*e^2 + e - 6)
29 1/2*(-e^2 + 4*e + 5)
81 3
389 1/2*(e^3 - 4*e^2 + 3*e - 8)
```

In the last step, we have give norms and the generators for the ideals which appear.

Exercise D.2. *Compute $\langle f, f \rangle$ for the elliptic curve of smallest conductor over $\mathbf{Q}(\sqrt{7})$.*

D.3. **Specifying the quaternion order.** In the project, we will want to identify a form which appears in several different guises (under the Jacquet-Langlands correspondence). For this, we will specify the quaternion order which is being used.

We continue to work over $F = \mathbf{Q}(\sqrt{5})$ but now we change the level. We work with newforms of level (21) and to do so we first define a quaternion algebra ramified at the primes (3), (7), and both real places.

```
> NN := ideal<Z_F | 3*7>;
> B := QuaternionAlgebra(NN, RealPlaces(F));
> B;
Quaternion Algebra with base ring F
> a, b := StandardForm(B);
> a, b;
-w - 5
42*w - 126
```

Thus we work with the quaternion algebra $B = \left(\frac{a,b}{F}\right) = \left(\frac{-w-5, 42w-126}{F}\right)$, i.e. the algebra generated by i, j subject to $i^2 = a$, $j^2 = b$, and $ji = -ij$. (The algorithm employed is probabilistic, so your a and b may be different.)

Now we compute a maximal order in this algebra and specify its use in the computation of cusp forms.

```
> O := MaximalOrder(B);
> M := HilbertCuspForms(F, NN);
> S := NewSubspace(M : QuaternionOrder := O);
> Dimension(S);
7
> newforms := NewformDecomposition(S);
> newforms;
[*
    New cuspidal space of Hilbert modular forms of dimension 1 over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
      Level = Ideal of norm 441 generated by ( [21, 0] )
      New at Ideal of norm 441 generated by ( [21, 0] )
      Weight = [ 2, 2 ],
    New cuspidal space of Hilbert modular forms of dimension 1 over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
      Level = Ideal of norm 441 generated by ( [21, 0] )
      New at Ideal of norm 441 generated by ( [21, 0] )
      Weight = [ 2, 2 ],
    New cuspidal space of Hilbert modular forms of dimension 1 over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
      Level = Ideal of norm 441 generated by ( [21, 0] )
      New at Ideal of norm 441 generated by ( [21, 0] )
      Weight = [ 2, 2 ],
```

```
    New cuspidal space of Hilbert modular forms of dimension 1 over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
      Level = Ideal of norm 441 generated by ( [21, 0] )
      New at Ideal of norm 441 generated by ( [21, 0] )
      Weight = [ 2, 2 ],
    New cuspidal space of Hilbert modular forms of dimension 3 over
      Number Field with defining polynomial x^2 - 5 over the Rational Field
      Level = Ideal of norm 441 generated by ( [21, 0] )
      New at Ideal of norm 441 generated by ( [21, 0] )
      Weight = [ 2, 2 ]
*]
```

We see there are 4 Hecke irreducible spaces of dimension 1 and one of dimension 3. Let's work with the one of dimension 3.

```
> f := Eigenform(newforms[5]);
> [[* Norm(pp), HeckeEigenvalue(f, pp) *] : pp in PrimesUpTo(30,F)];
[ [* 4,
    e
*], [* 5,
    -e + 1
*], [* 9,
    -1
*], [* 11,
    2
*], [* 11,
    2
*], [* 19,
    1/2*(-e^2 + 5)
*], [* 19,
    1/2*(-e^2 + 5)
*], [* 29,
    -2*e
*], [* 29,
    -2*e
*] ]
> H<e> := CoefficientField(f);
> H;
Number Field with defining polynomial x^3 - x^2 - 13*x + 5 over the Rational
Field
> Factorization(InnerProductVec(f,f));
[
```

```
    <Prime Ideal
    Two element generators:
        [2, 0, 0]
        [1, 1, 0], 8>,
    <Prime Ideal
    Two element generators:
        [5, 0, 0]
        [7, 8, 4], 1>,
    <Prime Ideal
    Two element generators:
        [37, 0, 0]
        [8, 2, 0], 1>
]
```

Now we observe this form in a related quaternion algebra, ramified only at the real places of F.

```
> B2 := QuaternionAlgebra(1*Z_F, RealPlaces(F));
> O2 := MaximalOrder(B2);
> S2 := NewSubspace(M : QuaternionOrder := O2);
> Dimension(S2);
7
> newforms2 := NewformDecomposition(S2);
> newforms2;
[*
    Cuspidal newform space of Hilbert modular forms over
        Number Field with defining polynomial x^2 - 5 over the Rational
    Field
        Level: Ideal of norm 441 generated by [21, 0]
        New level: Ideal of norm 441 generated by [21, 0]
        Weight: [ 2, 2 ]
        Dimension 1,
    Cuspidal newform space of Hilbert modular forms over
        Number Field with defining polynomial x^2 - 5 over the Rational
    Field
        Level: Ideal of norm 441 generated by [21, 0]
        New level: Ideal of norm 441 generated by [21, 0]
        Weight: [ 2, 2 ]
        Dimension 1,
    Cuspidal newform space of Hilbert modular forms over
        Number Field with defining polynomial x^2 - 5 over the Rational
    Field
```

```
        Level: Ideal of norm 441 generated by [21, 0]
        New level: Ideal of norm 441 generated by [21, 0]
        Weight: [ 2, 2 ]
        Dimension 1,
    Cuspidal newform space of Hilbert modular forms over
        Number Field with defining polynomial x^2 - 5 over the Rational
    Field
        Level: Ideal of norm 441 generated by [21, 0]
        New level: Ideal of norm 441 generated by [21, 0]
        Weight: [ 2, 2 ]
        Dimension 1,
    Cuspidal newform space of Hilbert modular forms over
        Number Field with defining polynomial x^2 - 5 over the Rational
    Field
        Level: Ideal of norm 441 generated by [21, 0]
        New level: Ideal of norm 441 generated by [21, 0]
        Weight: [ 2, 2 ]
        Dimension 3
*]
```

We identify the form f again as a constitutent in the unique irreducible space of dimension 3. But just to make sure, let's check the Hecke eigenvalues.

```
> f2 := Eigenform(newforms2[5]);
> [[* Norm(pp), HeckeEigenvalue(f2, pp) *] : pp in PrimesUpTo(30,F)];
[ [* 4,
    e
*], [* 5,
    -e + 1
*], [* 9,
    -1
*], [* 11,
    2
*], [* 11,
    2
*], [* 19,
    1/2*(-e^2 + 5)
*], [* 19,
    1/2*(-e^2 + 5)
*], [* 29,
    -2*e
*], [* 29,
```

```
    -2*e
*] ]
> [MinimalPolynomial(HeckeEigenvalue(f2,pp)) eq
> MinimalPolynomial(HeckeEigenvalue(f,pp)) : pp in PrimesUpTo(30,F)];
[ true, true, true, true, true, true, true, true, true ]
```

Now we compute the inner product.

```
> Factorization(InnerProductVec(f2,f2));
[
    <Prime Ideal
    Two element generators:
        [2, 0, 0]
        [1, 1, 0], 9>,
    <Prime Ideal
    Two element generators:
        [5, 0, 0]
        [1, 2, 0], 2>,
    <Principal Prime Ideal
    Generator:
        [3, 0, 0], 1>,
    <Prime Ideal
    Two element generators:
        [37, 0, 0]
        [8, 2, 0], 1>
]
```

We observe that $\langle f, f\rangle_1 \mid \langle f, f\rangle_2$. The project concerns exactly such relations between these factorizations.

Exercise D.3. *Match up the one-dimensional forms in the above spaces by their Hecke eigenvalues. How do the factorizations of $\langle f, f\rangle_B$ compare for these forms for the different quaternion algebras B?*

Exercise D.4. *Let $\mathfrak{N}$ be the product of the first 4 primes over $\mathbf{Q}(\sqrt{5})$. Compute the norm $\langle f, f\rangle_B$ for each newform of level $\mathfrak{N}$ in each of its quaternionic manifestations, i.e. for each quaternion algebra B of discriminant dividing $\mathfrak{N}$ (which will necessarily be composed of an even number of prime factors).*

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