K. Prasanna March 15, 2011
(Lecture 4)

- $$\begin{array}{c} CH^i(X)\otimes \underline{\underline{\mathbb{Q}_\ell}} \\ \cup\!\mid \\ CH^i(X)\otimes \mathbb{Q}. \end{array}$$

Integral Hodge/Tate conj are not expected to be true.
versions of

- Why do integral period relations?
- $\underline{\underline{Q}}$: Can we bound denominators in Tate conj?

- Theta lifts.
  - Nonvanishing of theta lifts?

    Can be subtle: $\left\{ \begin{array}{l} \text{local conditions: } \varepsilon\text{-factors} \\ \text{global conditions: L-value.} \end{array} \right.$
  - Algebraicity / Integrality? ⟷ Iwasawa thy.
  - Lift is a p-unit?

F = tot real.

$\left\{\begin{array}{ll} \text{Harris:} & \text{Unitary groups.} \\ \text{Ichino/P:} & \text{Quaternionic unitary gps.} \end{array}\right.$

<u>Harris.</u> : B B'

- E/F CM field, $E \hookrightarrow B$, $E \hookrightarrow B'$

$$\begin{tikzcd} B \arrow[r, bend left, "\mathrm{pr}"] & E + Ej = E + jE \end{tikzcd}$$

$\langle x, y\rangle = xy^{i} + yx^{i}$,

i = main involution.

Think of B as an right.. E-vector space.

Can make B into a unitary space:

$\underline{\begin{array}{c} \alpha j = j\bar{\alpha} \\ \alpha \in E \end{array}}$.

$$\langle\langle x, y\rangle\rangle = \mathrm{pr}(x^{i} \cdot y), \quad \mathrm{tr}\,\langle\langle\,,\,\rangle\rangle = \langle\,,\,\rangle.$$

$\langle\langle\,,\,\rangle\rangle$ is an E-Hemitian form.

$$\begin{tikzcd} GU_E(B) \arrow[r, hook] \arrow[d, equal] & GO(B), \arrow[d, equal] \\ (B^\times \times E^\times)/F^\times & (B^\times \times B^\times)/F^\times \end{tikzcd}$$

A form on $GU_E(B)$ is a pair (π, χ), π form on $B^\times$

χ Hecke char of $E^\times$.

$$\xi_\pi \cdot \xi_\chi = 1 .$$

B, B' $\quad U_E(B) \overset{\text{theta lift}}{\rightsquigarrow} U_E(B')$.

$$\begin{tikzcd} & 0 \\ (\pi,\chi) \arrow[ur] \arrow[r] & (\pi^\vee, \chi^{-1}) \end{tikzcd}$$

$$\begin{tikzcd} GL_2 & (B^\times \times B^\times)/F^\times \arrow[l] \arrow[l, shift left, leftarrow] \end{tikzcd}$$

Harris studies arithmetic of this theta lift .

- ε-factors
- $L(\frac{1}{2}, \pi, \chi)$ (Rankin-Selberg) .
- Rallis inner product formula .

$$\begin{tikzcd} & & \in \overline{\mathbb{Q}} . \\ & f_{B'} \times \chi^{-1} \arrow[ur, phantom, "\textcircled{\alpha}"] & \\ f_B \times \chi \arrow[ur] & & \end{tikzcd}$$

$$\begin{tikzcd} & U(B') \\ U(B^0) \arrow[ur] & \end{tikzcd}$$

$$\boxed{\langle \underset{\chi}{\theta_\varphi(f_B)}, \theta_\varphi(f_{B'}\chi)\rangle = L(\tfrac{1}{2}, \pi, \chi) \cdot \langle f_B, f_B \rangle}$$

$\searrow GU_E(B')$

- If ~~B~~ B is atleast as ramified as B' at ∞, then this lift is algebraic.

$B' = E + Ej' = V_1 + V_2$

$$\begin{tikzcd} U(B)\times U(B) \arrow[dr, dash] & U(B') \arrow[d, dash] \\ U(B) \arrow[ur, dash] & U(V_1)\times U(V_2) \end{tikzcd}$$

$U(V_1) = U(1)$, $U(V_2) = U(1)$

To understand $U(1) \to U(B)$, Harris again uses Rallis .

Joint / Ichino :

- Period integrals to L-values.

(Waldspurger / Tunnell-Saito) .

F, π on $GL_2(\mathbb{A}_F)$, E/F , χ a Hecke char. of E

$$\boxed{\xi_\pi \cdot \xi_\chi = 1 .}$$

B quat. $alg/_F$ such that π transfers to π_B on $B^\times$.

$f_B \in \pi_B$, suppose $E \hookrightarrow B$.

$$\int f_B \big|_{E^\times_{\mathbb{A}}} \cdot \chi \qquad \qquad \text{sign of } \underset{\overset{\shortparallel}{}}{L(s,f,\chi)}$$

<u>Thm:</u> (Tun./Saito/Wald). Given, $\pi, \overset{E}{\chi}$, , such that $\varepsilon(\frac{1}{2},\pi,\chi) = +1$,

$\exists$ a unique quat. alg B, such that π transfers to π_B on $B^\times$,

& $\exists f_B \in \pi_B$, $\left[\int f_B \cdot \chi\right]^2 \doteq L(\frac{1}{2},\pi,\chi)$. (Normalization factors).

$\varepsilon_v(B) = \xi_{\pi,v}(-1) \cdot \varepsilon_v(\frac{1}{2},\pi,\chi)$

- Triple Product. ($\overset{\text{Garrett}}{\text{Harris-Kudla}}$ /Watson / Ichino-Ikeda).

π_1, π_2, π_3 on $GL_2(\mathbb{A}_F)$.

$\xi_{\pi_1} \cdot \xi_{\pi_2} \cdot \xi_{\pi_3} = 1$. $\quad L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$

Suppose $\varepsilon(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) = +1$. $\quad \varepsilon_v(B) = \varepsilon_v(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$.

Then $\exists$ a unique quaternion algebra B such that

π_1, π_2, π_3 transfer to $\pi_1^B, \pi_2^B, \pi_3^B$, & forms f_1^B, f_2^B, f_3^B

s.t.

$$\left[\int f_1^B \cdot f_2^B \cdot f_3^B \right]^2 \doteq L(\tfrac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3).$$

- Are these two formulae related? $\qquad \begin{array}{c} E \\ |{\scriptstyle 2} \\ F \end{array} \circlearrowleft \rho.$

$\pi_1 = \pi$, E/F CM, η_1, η_2 Hecke chars of $\textcircled{E}$.

$\pi_2 = \pi\eta_1$, $\pi_3 = \pi\eta_2$. $\quad \boxed{\xi_\pi \cdot \xi_{\eta_1} \cdot \xi_{\eta_2} = 1}$

$$\underset{\substack{B \quad \overset{+1}{\parallel} \\ [\int]^2}}{L(s, \pi \otimes \pi\eta_1 \otimes \pi\eta_2)} = \underset{\substack{B_1 \quad \overset{+1}{\parallel} \\ [\int]^2}}{L(s, \pi, \chi_1)} \cdot \underset{\substack{\overset{+1}{\parallel} \quad B_2 \\ [\int]^2}}{L(s, \pi, \chi_2)} \quad \begin{cases} \chi_1 = \eta_1 \eta_2 \\ \chi_2 = \eta_1 \eta_2^\rho \end{cases}$$

$B = B_1 \cdot B_2$ in Brauer gp.

$E \hookrightarrow B_1, B_2$. $\Rightarrow E \hookrightarrow B$

$B_1 = E + Ej_1$, $B_2 = E + Ej_2$, $\mathrm{Tr}(j_1) = \mathrm{Tr}(j_2) = 0$

$J_1 := j_1^2 \in F$, $J_2 := j_2^2 \in F$.

Think of B_1, B_2 as right E-spaces. (unitary)

$\langle , \rangle_1, \langle , \rangle_2$.

$B = E + Ej$, $j^2 = J_1 J_2$.

$V = B_1 \otimes_E B_2 \circlearrowleft B$

4-dim as E-space

$$(1 \otimes 1) \cdot j = j_1 \otimes j_2$$
$$(j_1 \otimes 1) \cdot j = J_1 \cdot (1 \otimes j_2)$$
$$(1 \otimes j_2) \cdot j = J_2 \cdot (j_1 \otimes 1)$$
$$(j_1 \otimes j_2) \cdot j = J_1 J_2 \cdot (1 \otimes 1).$$

check this gives an action.

Pick $\alpha \in E$, $\mathrm{tr}_{E/F}(\alpha) = 0$, $\alpha \neq 0$

is a 2-dim right B-space. $\langle , \rangle = \alpha \langle , \rangle_1 \otimes \langle , \rangle_2$ on $B_1 \otimes_E B_2$

$\downarrow$

skew-Hermitian form

Can find a B-skew Hermitian form $\langle\langle , \rangle\rangle$ on V, s.t.

$$\mathrm{pr}\circ\langle\langle , \rangle\rangle = \langle , \rangle .$$

$$\langle\langle x\alpha, y\beta \rangle\rangle = \alpha^{\iota} \langle\langle x,y \rangle\rangle \cdot \beta , \qquad \alpha, \beta \in B^{\times} .$$

$$\langle\langle x,y \rangle\rangle = -\langle\langle y,x \rangle\rangle^{\iota} \qquad \text{(skew- Hemitian)} .$$

$$GU_B(V)^0 = (B_1^{\times} \times B_2^{\times})/F^{\times}$$

$W = B$, usual B-Hermitian form, $\langle x,y \rangle = xy^{\iota}$.

$$GU_B(W) = B^{\times}$$

$$\begin{aligned} GU_B(W) &\longrightarrow GU_B(V)^0 \\ B^{\times} &\longrightarrow (B_1^{\times} \times B_2^{\times})/F^{\times} . \end{aligned}$$

Thm: $\Theta(\pi_B) = (\pi_{B_1} \times \pi_{B_2})$.

$$f_B \longrightarrow f_{B_1} \times f_{B_2}.$$

$$\begin{tikzcd}
B^\times \times B^\times \arrow[d, dash] \arrow[dr, dash] & (B_1^\times \times B_2^\times)/F^\times \arrow[d, dash] \\
B^\times \arrow[ur, dash] & (E^\times \times E^\times)/F^\times
\end{tikzcd}$$

See-saw duality $\Rightarrow$ equality of periods.

$$f_B \longrightarrow \alpha(B_1, B_2)\, f_{B_1} \times f_{B_2}$$

Rallis: $\alpha(B_1,B_2)^2 \cdot \langle f_{B_1}, f_{B_1}\rangle \cdot \langle f_{B_2}, f_{B_2}\rangle = \langle f_B, f_B\rangle \cdot L(1, \mathrm{ad}^0\pi)$

$$\alpha(B_1, B_2)^2 \cdot \frac{L(1, \mathrm{ad}^0\pi)}{\prod_{v\in\Sigma_{B_1}} c_v} \cdot \frac{L(1, \mathrm{ad}^0\pi)}{\prod_{v\in\Sigma_{B_2}} c_v} = \frac{L(1, \mathrm{ad}^0\pi)}{\prod_{v\in\Sigma_B} c_v} \cdot L(1, \mathrm{ad}^0\pi)$$

$$\alpha(B_1,B_2)^2 = \frac{\prod_{v\in\Sigma_{B_1}} c_v \cdot \prod_{v\in\Sigma_{B_2}} c_v}{\prod_{v\in\Sigma_B} c_v} = \prod_{v\in\Sigma_{B_1}\cap\Sigma_{B_2}} c_v^2$$

$$\alpha(B_1,B_2) = \prod_{v\in\Sigma_{B_1}\cap\Sigma_{B_2}} c_v.$$

$$\left.\begin{array}{l} \underline{\text{Conj}}: B \text{ (i) If } \Sigma_{B_1,\infty} \cap \Sigma_{B_2,\infty} = \phi, \text{ then } \underline{\alpha(B_1,B_2) \in \overline{\mathbb{Q}}}. \\ \qquad\qquad \& \text{ is a } p\text{-integer}. \\ \text{(ii) If further } \Sigma_{B_1} \cap \Sigma_{B_2} = \phi, \text{ then } \alpha(B_1,B_2) \\ \qquad\qquad \text{is a } p\text{-unit}. \end{array}\right\}$$

Conj B ⇒ Conj A :

We need to define c_v, $v \in \Sigma(\pi)$.

Let $S \subseteq \Sigma(\pi)$, $\# S$ even : $c_S := \dfrac{L(1, \mathrm{ad}^\circ \pi)}{\langle f_S, f_S \rangle}$

Conj B $\Rightarrow$ $c_{S \sqcup T} = c_S \cdot c_T$.

To define c_v: Pick r, s two other places in $\Sigma(\pi)$.

$$c_v^2 = \frac{c_{vr}\, c_{vs}}{c_{rs}}$$

$$\frac{\cancel{c_{vr}}\, c_{vt}}{c_{rt}} = \frac{\cancel{c_{vr}}\, c_{vs}}{c_{rs}} \qquad \underline{c_{vt} \cdot c_{rs} = c_{vs} \cdot c_{rt}}$$

Easy to see $c_S = \prod_{v \in S} c_v$.