

$f \in S_2(\Gamma_0(N))$ newform. $\leftrightarrow E$.

B quat alg., indet. / \mathbb{Q} of disc $N^+ | N^-$.

$$N = N^+ \cdot N^-$$

X_B associated to an (Eichler) order of level N^+ in B .

$\langle f_B, f_B \rangle$ $\langle f, f \rangle$. (f_B, f normalized up to λ -adic units).

Assume $\bar{\rho}_{f, \lambda}$ is irreducible. (λ is not Eisenstein).

Prop.

$$\frac{\langle f, f \rangle}{\langle f_B, f_B \rangle} \underset{\lambda}{\sim} \frac{\prod_{\mathfrak{p} | N^+} C_{\mathfrak{p}}}{\prod_{\mathfrak{p} | N^-} C_{\mathfrak{p}}} \quad (\text{up to } \lambda\text{-adic units}).$$

True for abelian variety quotients.

$$\langle f_B, f_B \rangle \underset{\lambda}{\sim} \frac{\langle f, f \rangle}{\prod_{\mathfrak{p} | N^+} C_{\mathfrak{p}}} = \frac{L(1, \text{ad}^* f)}{\prod_{\mathfrak{p} | N^+} C_{\mathfrak{p}}}$$

$F \ni$ tot real.

$\exists C_{v_i} \rightarrow C_{v_d}$, s.t.

$$\langle f_B, f_B \rangle \sim \prod_{\substack{B\text{-split} \\ \text{at } v_i}} C_{v_i} \sim \frac{\prod C_{v_i}}{\prod_{\substack{\text{Bram} \\ \text{at } v_i}} C_{v_i}} \sim \frac{L(1, \text{ad}^0 f)}{\prod_{\substack{\text{Bram} \\ \text{at } v_i}} C_{v_i}}$$

Assume \mathcal{A} is not Eis-fnf.

Conj: \exists a function: $c: \sum_{v \rightarrow C_v} (\pi) \rightarrow \mathbb{C}^X$, s.t.

$$\langle f_B, f_B \rangle \sim_{\lambda\text{-units}} \frac{L(1, \text{ad}^0 f)}{\prod_{v \in \Sigma_B} C_v}$$

- If v is inf, expect C_v are transcendental, & alg. ind. except if f is a Basechange.
- If v is finite, expect C_v are (λ -adic) integers, & count level-lowering congruences.

Recall Conjecture:

F tot real, f HM newform, $\pi = \text{aut rep}$

\exists invariants c_v , $v \in \Sigma(\pi)$, such that

$$\langle f_B, f_B \rangle = \frac{L(1, \text{ad}^0 \pi)}{\prod_{v \in \Sigma(B)} c_v} \quad (\text{upto Eis primes})$$

- If v is infinite, $c_v = \text{transcendental}$
- If v is finite, $c_v = \text{algebraic integer}$

(if $\prod c_v$, then $f \equiv g \pmod{p}$, $\forall X$ level (g))

Notes: Thm: Suppose $F = \mathbb{Q}$, $f \leftrightarrow$ isogeny class of elliptic curves

$$f \in S_2(\Gamma_0(N)), N \text{ square-free.}$$

What are the c_v 's? $\Sigma(\pi) = \{\infty\} \cup \{q \mid q \mid N\}$.

$$c_\infty = \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_E}$$

E any elliptic curve in isogeny class, $\omega_E = \text{Neron differential}$

For $q \mid N$, $c_q = \text{order of component gp of Neron model of } E \text{ at } q$.

If E' is another elliptic curve in same isogeny class, $E \rightarrow E'$

B definite. ($X_B =$ finite set of pts).

$$\langle f_B, f_B \rangle = \frac{L(1, ad)}{C_\infty \cdot \prod_{q|d_B} C_q}$$

There are relations between $\langle f_B, f_B \rangle$, as B varies.

(Student Project 1: Compute this for totally definite q-algebras ~~of rank~~)

- higher weight
- tot real fields .

Theta correspondence :

F = number field, \mathbb{A}_F ; W a symplectic space over F .

$\langle, \rangle : W \times W \rightarrow F$ that is nondegenerate,
& alternating.

Fix ψ an additive character of $F \backslash \mathbb{A}_F$.

Let $Sp(W)$ be the symplectic gp of W .

($GSp(W)$: similitude gp)

$$1 \rightarrow \mathbb{C}^* \rightarrow \underbrace{Mp(W)(\mathbb{A})}_{\psi} \rightarrow Sp(W)(\mathbb{A}) \rightarrow 1$$

Weil Representation: $\omega_\psi : Mp(W)(\mathbb{A}) \rightarrow \text{Aut}(\mathcal{S})$

Dual Reductive pair: (Howe)

(G_1, G_2) of reductive gps, $G_1 \times G_2 \subseteq Sp(W)$,

G_1 & G_2 are centralizers of each other in $Sp(W)$.

$$\begin{array}{ccc}
 G_1 \times G_2 \subseteq Sp(W) & & \\
 \widetilde{G}_1(A) \times \widetilde{G}_2(A) \subseteq \boxed{MP_\psi(W(A))} & \xrightarrow{\quad} & W_\psi \text{ (Weil)} \\
 \downarrow \quad \downarrow & & \downarrow \\
 G_1(A) \times G_2(A) \subseteq Sp(W)(A) & &
 \end{array}$$

Can use this to transfer functions from $G_1(A)$ to $G_2(A)$ & in the other direction.

Eg. 1. $W =$ symplectic space, $V =$ orthogonal space.
 $W \otimes V$ is a symplectic space.
 $(Sp(W), O(V))$ is a dual reductive pair in $Sp(W)$.

Eg. 2. K quadratic extn. V_1, V_2 unitary spaces / K .
 F \downarrow Hermitian \rightarrow skew-Hermitian

V_1 : K -vector space, $\langle, \rangle : V_1 \times V_1 \rightarrow K$
 $\langle \alpha x, \beta y \rangle = \bar{\alpha} \langle x, y \rangle \beta$
 $\langle x, y \rangle = \overline{\langle y, x \rangle}$

V_2 : $\langle x, y \rangle = -\overline{\langle y, x \rangle}$

$W = V_1 \otimes_K V_2$ thought of as an F -vector space.

$\langle, \rangle = \text{tr}_{K/F} (\langle, \rangle_1 \otimes \langle, \rangle_2)$ skew-symmetric.

$(U_K(V_1), U_K(V_2)) \subseteq \text{Sp}(W)$ is a dual reductive pair.

We can use Weil rep to construct an integrating kernel.

$$\varphi \in \mathcal{L} \rightsquigarrow \Theta_\varphi(g_1, g_2)$$

$(G_1, G_2) \subseteq \text{Sp}(W)$.

f_1 on $G_1(A)$: $\Theta_\varphi(A)^{(g_2)} = \int f_1(g_1) \cdot \underbrace{\Theta_\varphi(g_1, g_2)}_{\text{kernel}} dg_1$.

Eg. Shimizu correspondence B quat alg / F .

$V = B$, $\langle x, y \rangle = xy^i + yx^i$, $i = \text{main involution}$.

$W = 2\text{-dim symplectic space}$

$(\text{Sp}(W), \text{O}(V))$ $(\text{GSp}(W), \text{GO}(V))$
" $(\text{GL}_2, (B^x \times B^x) / F^x)$

$F^x \backslash B^x \times B^x \rightarrow \text{GO}(V)^0$ $(\alpha, \beta) \mapsto (x \mapsto \alpha x \beta^{-1})$

Forms in $(B_1^x \times B_2^x) / F^x$ look like pairs (π_1, π_2) ,

S.t. $\omega_{\pi_1} \cdot \omega_{\pi_2} = 1$.

↓
central chars.

π on $GL_2(A)$; $\theta(\pi) = \begin{cases} 0 & \text{if } \pi \text{ doesn't transfer to } B^x \\ \pi_B \times \pi_B^v, & \pi_B = JL(\pi) \end{cases}$

In our case, central chars trivial $\Rightarrow \pi_B^v \cong \pi_B$.

Pick $f \in \pi$, $\theta_\varphi(f) = \alpha(f_B \times f_B)$ (can pick φ to make this happen).

$GL_2 \rightarrow (B^x \times B^x) / F^x$

$GL_2 \leftarrow (B^x \times B^x) / F^x$ (Easier to study)

$\beta \cdot f \leftarrow f_B \times f_B$

since you can compute explicitly with F.C.'s on left.

One can show; $\beta = \langle f_B, f_B \rangle$.

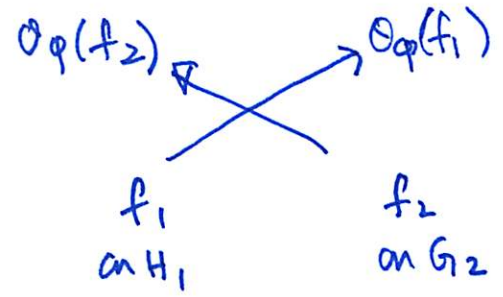
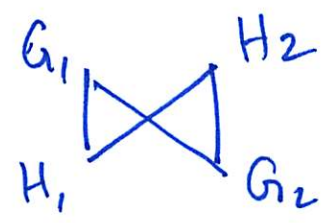
$\alpha \langle f_B \times f_B, f_B \times f_B \rangle = \beta \cdot \langle f, f \rangle = \langle f_B, f_B \rangle \langle f, f \rangle$

$\alpha = \frac{\langle f, f \rangle}{\langle f_B, f_B \rangle}$

Seesaw Dual Reductive Pair (Kudla)

$$(G_1, G_2), (H_1, H_2) \subseteq \text{Sp}(W)$$

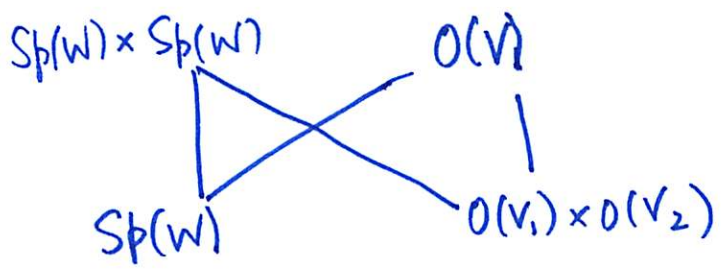
These form a seesaw.

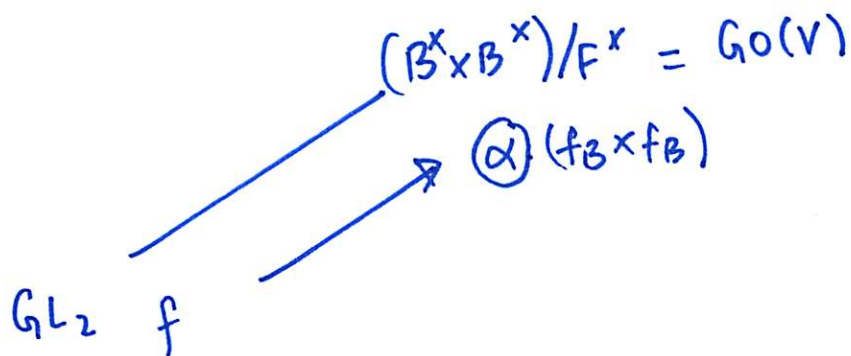


$$\int f_1 \cdot O_\phi(f_2)|_{H_1} = \int O_\phi(f_1)|_{G_2} \cdot f_2 \quad \bullet \text{ seesaw duality.}$$

Eg.: V orthogonal, W symplectic. $\mathbb{W} = V \otimes W$.
 $(O(V), \text{Sp}(W)) \subseteq \text{Sp}(\mathbb{W})$

$V = V_1 \oplus V_2$ (sum of two orthogonal spaces)





F = Q., B indefinite.

Form on $B^x \rightsquigarrow$ section of a line bundle on X_B .

(Usual Modular forms: function on pairs (E, ω) .)

X_B : coarse moduli space, abelian surfaces with end. by B.

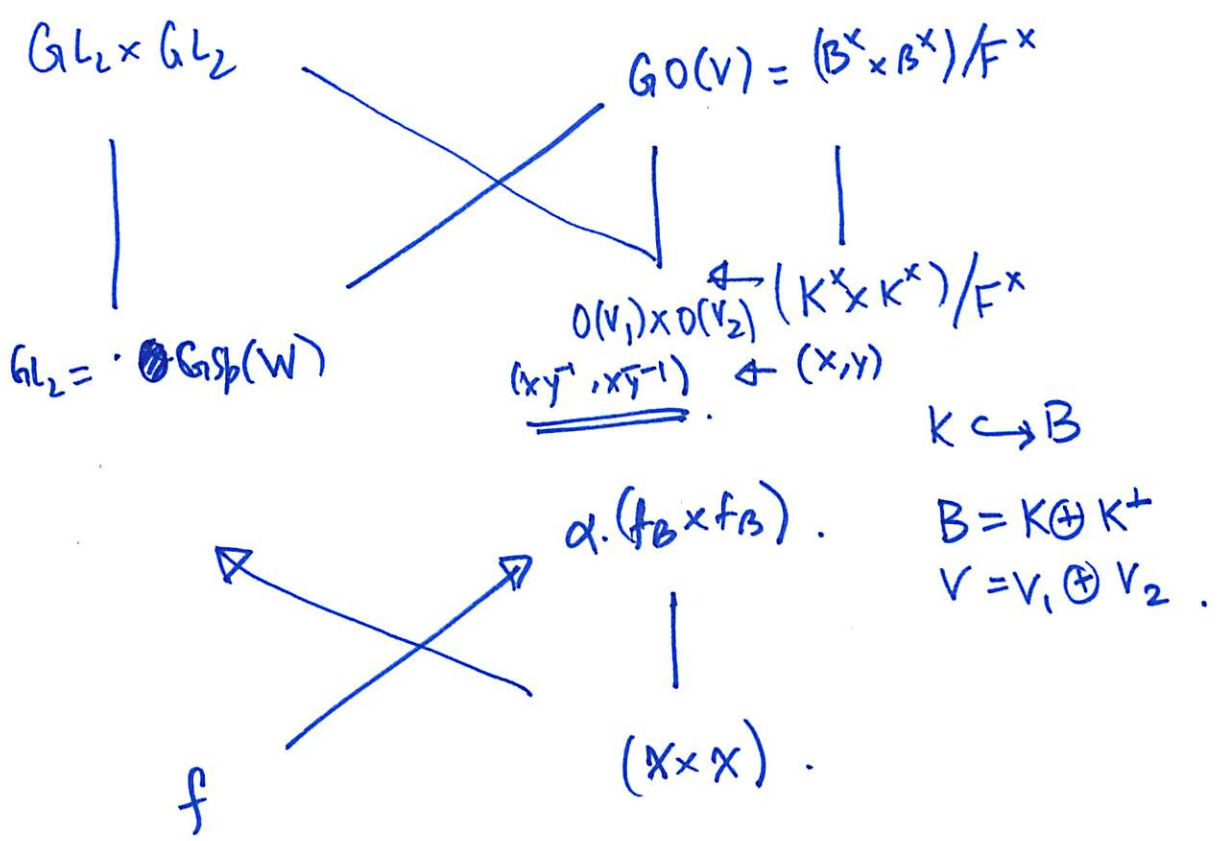
Check this on CM points; $K \hookrightarrow B, K^x \hookrightarrow B^x$
 \downarrow \downarrow
 g g

$$L_x(g) = \int_{K^x/A} g \cdot X$$

Pick a Hecke char X of K of int type $(2,0)$
 \hookrightarrow weight.

= "finite sum of values of g , twisted by X ".

Criterion: g is rational (integral) if $L_x(g)$ are rational (integral) up to periods of CM elliptic curves. (CM periods).



$$\alpha \cdot \int (f_B \times f_B) \cdot (X \times X) = \int \theta(X \times 1) |_{GL_2} \cdot f$$

$$= \int f \cdot \overline{\theta(x)} \cdot \theta(1)$$

$$\textcircled{\alpha} \frac{L_X(f_B)^2}{\Omega_{CM}} = \frac{\langle f \theta(1), \theta(x) \rangle}{\Omega_{CM}} = \text{value at } s=1/2$$

value of Eis. series: $E(s)$ or $L(s, f, X)$

integral rep for $L(s, f, X)$

- Harris Kudla: L-value is rational.
- P. (2003). L-values are integral (use Main conj. for imag. quad fields. (Rubin)).
- Factorization: (p-adic families)...