K. Prasanna March 14, 2011 (Lec. 3)

$f \in S_2(\Gamma_0(N))$ newform. $\longleftrightarrow E$.

B quat alg., indef. $/\mathbb{Q}$ of disc $N^-|N$.

$N = N^+ \cdot N^-$

X_B associated to an (Eichler) order of level N^+ in B.

$\langle f_B, f_B \rangle$ $\langle f, f \rangle$. (f_B, f normalized up to ℓ-adic units).

Assume $\bar{\rho}_{f,\lambda}$ is irreducible. (λ is not Eisenstein).

<u>Prop:</u> $$\frac{\langle f, f \rangle}{\langle f_B, f_B \rangle} \sim_{\lambda} \prod_{\underbrace{p | d_B}_{p|N^-}} c_p$$ (up to λ-adic units).

True for abelian variety quotients.

$$\langle f_B, f_B \rangle \sim_{\lambda} \frac{\langle f, f \rangle}{\prod_{p|d_B} c_p} = \frac{L(1, \mathrm{ad}^0 f)}{\prod_{p|d_B} c_p}$$

F = tot real.

$\exists\ c_{v_1}, \dots, c_{v_d}$, s.t.

$$\langle f_B, f_B\rangle \sim \prod_{\substack{B \text{ split}\\ \text{at } v_i}} c_{v_i} \sim \frac{\prod c_{v_i}}{\prod\limits_{\substack{B \text{ ram}\\ \text{at } v_i}} c_{v_i}} \sim \frac{L(1, \mathrm{ad}^0 f)}{\prod\limits_{\substack{B \text{ ram}\\ \text{at } v_i}} c_{v_i}}$$

Assume λ is not Eis. prif.

<u>Conj:</u> $\exists$ a function: $c : \Sigma(\pi) \longrightarrow \mathbb{C}^\times$, s.t.

$$v \longmapsto c_v$$

$$\langle f_B, f_B\rangle \underset{\lambda\text{-units}}{\sim} \frac{L(1, \mathrm{ad}^0 f)}{\prod\limits_{v \in \Sigma_B} c_v}$$

- If v is inf, expect c_v are transcendental, & alg. ind. except if f is a Base change.

- If v is finite, expect c_v are (λ-adic) integers, & count level-lowering congruences.

Recall Conjecture:

F tot real, f HM newform, π = aut rep

$\exists$ invariants c_v, $v \in \Sigma(\pi)$, such that

$$\langle f_B, f_B \rangle = \frac{L(1, \mathrm{ad}^0 \pi)}{\prod_{v \in \Sigma(B)} c_v}$$ (upto Eis primes)

- If v is infinite, c_v = transcendental
- If v is finite, c_v = algebraic integer

(if $p | c_v$, then $f \equiv g$ (p), $v \nmid$ level(g))

Notes: Thm: Suppose $F = \mathbb{Q}$, $f \leftrightarrow$ isogeny class of elliptic curves

$f \in S_2(\Gamma_0(N))$, N square-free.

What are the c_v's? $\Sigma(\pi) = \{\infty\} \cup \{q \mid q | N\}$.

$$c_\infty = \int_{E(\mathbb{C})} \omega_E \wedge \bar{\omega}_E$$

E any elliptic curve in isogeny class, ω_E = Neron differential

For $q | N$, c_q = order of component gp of Neron model of E at q.

If E' is another elliptic curve in same isogeny class, $E \dashrightarrow E'$

B definite. (X_B = finite set of pts).

$$\langle f_B, f_B \rangle = \frac{L(1, \mathrm{ad})}{c_\infty \cdot \prod_{q \mid d_B} c_q}$$

There are relations between $\langle f_B, f_B \rangle$, as B varies.

(Student Project 1: Compute this for totally definite q-algebras $\neq$ ~~[illegible]~~)

- higher weight
- tot real fields.

Theta correspondence :

F = number field, $\mathbb{A}_F$; $\mathbb{W}$ a symplectic space over F.

$\langle , \rangle : \mathbb{W} \times \mathbb{W} \longrightarrow F$ that is nondegenerate, & alternating.

Fix ψ an additive character of $F \backslash \mathbb{A}_F$.

Let $Sp(\mathbb{W})$ be the symplectic gp of $\mathbb{W}$.

($GSp(\mathbb{W})$: similitude gp)

$$1 \to \mathbb{C}^* \to \underbrace{Mp(\mathbb{W})(\mathbb{A})}_{\psi} \to Sp(\mathbb{W})(\mathbb{A}) \to 1$$

Weil Representation: $\omega_\psi : Mp_\psi(\mathbb{W})(\mathbb{A}) \longrightarrow Aut(\mathcal{S})$

Dual Reductive pair: (Howe)

(G_1, G_2) of reductive gps, $G_1 \times G_2 \subseteq Sp(\mathbb{W})$,

G_1 & G_2 are centralizers of each other in $Sp(\mathbb{W})$.

$$
\begin{array}{ccc}
G_1 \times G_2 & \subseteq & Sp(\mathbb{W}) \\
\widetilde{G_1}(\mathbb{A}) \times \widetilde{G_2}(\mathbb{A}) & \subseteq & \boxed{Mp_\psi(\mathbb{W}(\mathbb{A}))} \longrightarrow \omega_\psi \text{ (Weil)}. \\
\downarrow \quad \downarrow & & \downarrow \\
G_1(\mathbb{A}) \times G_2(\mathbb{A}) & \subseteq & Sp(\mathbb{W})(\mathbb{A})
\end{array}
$$

Can use this to transfer functions from $G_1(\mathbb{A})$ to $G_2(\mathbb{A})$ & in the other direction.

Eg. 1. W = symplectic space, V = orthogonal space.

$\mathbb{W} = W \otimes V$ is a symplectic space.

$(Sp(W), O(V))$ is a dual reductive pair in $Sp(\mathbb{W})$.

Eg. 2. $\begin{array}{c} K \\ G \,|\, \text{quadratic extn.} \\ F \end{array}$ V_1, V_2 unitary spaces $/K$.

$V_1 \to$ Hermitian, $V_2 \to$ Skew-Hermitian

V_1: K-vector space, $\langle\,,\,\rangle : V_1 \times V_1 \longrightarrow K$

$$\langle \alpha x, \beta y \rangle = \bar{\alpha} \langle x, y \rangle \beta$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

V_2: $\langle x, y \rangle = -\overline{\langle y, x \rangle}$.

$W = V_1 \otimes_K V_2$ thought of as an F-vector space.

$\langle , \rangle = \mathrm{tr}_{K/F} \left(\langle , \rangle_1 \otimes \langle , \rangle_2 \right)$ skew-symmetric.

$\left(U_K(V_1), U_K(V_2) \right) \subseteq Sp(W)$ is a dual reductive pair.

We can use Weil rep to construct an integrating kernel.

$$\varphi \in \mathcal{S} \rightsquigarrow \theta_\varphi(g_1, g_2)$$

$(G_1, G_2) \subseteq Sp(W)$.

f_1 on $G_1(\mathbb{A})$: $\quad \theta_\varphi(f_1)^{(g_2)} = \int f_1(g_1) \cdot \underbrace{\theta_\varphi(g_1, g_2)}\, dg_1$.

Eg. (Shimizu correspondence) B quat alg$/F$.

$V = B$, $\langle x, y \rangle = x y^i + y x^i$, i = main involution.

W = 2-dim symplectic space

$(Sp(W), O(V)) \qquad (GSp(W), GO(V))$

$\qquad\qquad\qquad\qquad\; (GL_2, (B^\times \times B^\times)/F^\times)$

$F^\times \backslash B^\times \times B^\times \longrightarrow GO(V)^0 \qquad (\alpha, \beta) \longmapsto (x \mapsto \alpha x \beta^{-1})$

Forms on $(B_1^\times \times B_2^\times)/F^\times$ look like pairs (π_1, π_2),

s.t. $\omega_{\pi_1} \cdot \omega_{\pi_2} = 1$.

central chars.

π on $GL_2(\mathbb{A})$; $\theta(\pi) = \begin{cases} 0 & \text{if } \pi \text{ doesn't transfer to } B^\times \\ \pi_B \times \pi_B^\vee, & \pi_B = JL(\pi). \end{cases}$

In our case, central chars trivial $\Rightarrow \pi_B^\vee \cong \pi_B$.

Pick $f \in \pi$, $\theta_\varphi(f) = \alpha\,(f_B \times f_B)$ (can pick φ to make this happen).

$GL_2 \longrightarrow (B^\times \times B^\times)/F^\times$

$GL_2 \longleftarrow (B^\times \times B^\times)/F^\times$ (Easier to study) since you can compute explicitly with F.C.'s on left.

$\beta \cdot f \longleftarrow f_B \times f_B$

One can show; $\beta = \langle f_B, f_B \rangle$.

$$\alpha \langle f_B \times f_B, f_B \times f_B \rangle = \beta \cdot \langle f, f \rangle = \langle f_B, f_B \rangle \langle f, f \rangle$$

$$\boxed{\alpha = \frac{\langle f, f \rangle}{\langle f_B, f_B \rangle}}$$

Seesaw Dual Reductive Pair (Kudla).

$(G_1, G_2),\ (H_1, H_2) \subseteq Sp(\mathbb{W})$.

These form a seesaw.

$$\begin{tikzcd} G_1 \arrow[d, dash] \arrow[dr, dash] & H_2 \arrow[d, dash] \arrow[dl, dash] \\ H_1 & G_2 \end{tikzcd} \qquad \begin{tikzcd} \Theta_\varphi(f_2) & \Theta_\varphi(f_1) \\ \begin{matrix} f_1 \\ \text{on } H_1 \end{matrix} \arrow[ur] & \begin{matrix} f_2 \\ \text{on } G_2 \end{matrix} \arrow[ul] \end{tikzcd}$$

$$\int f_1 \cdot \Theta_\varphi(f_2)\big|_{H_1} = \int \Theta_\varphi(f_1)\big|_{G_2} \cdot f_2$$ • seesaw duality.

<u>Eg</u>: V orthogonal, W symplectic. $\mathbb{W} = V \otimes W$.

$(O(V), Sp(W)) \subseteq Sp(\mathbb{W})$

$V = V_1 \oplus V_2$ (sum of two orthogonal spaces).

$$\begin{tikzcd} Sp(W) \times Sp(W) \arrow[d, dash] \arrow[dr, dash] & O(V) \arrow[d, dash] \arrow[dl, dash] \\ Sp(W) & O(V_1) \times O(V_2) \end{tikzcd}$$

$$(B^\times \times B^\times)/F^\times = GO(V)$$

$$\alpha(f_B \times f_B)$$

$$GL_2 \quad f$$

$\underline{\underline{F=\mathbb{Q}}}$: , B indefinite.

Form on $B^\times \rightsquigarrow$ section of a line bundle on X_B.

(Usual Modular forms: function on pairs (E, ω).).

X_B: coarse moduli space, abelian surfaces with end. by B.).

Check this on CM points; $K \hookrightarrow B$, $K^\times \hookrightarrow B^\times$

g

$$L_\chi(g) = \int_{K^\times_{\mathbb{A}}} g \cdot \chi$$

Pick a Hecke char χ of K of inf type $(2,0)$

↳ $\underline{\underline{\text{weight}}}$.

= "finite sum of values of g, twisted by χ".

<u>Criterion:</u> g is rational (integral) if $L_\chi(g)$ are rational (integral) up to periods of CM elliptic curves. (CM periods).

$GL_2 \times GL_2$ $\qquad$ $GO(V) = (B^\times \times B^\times)/F^\times$

$GL_2 = GSp(W)$ $\qquad$ $O(V_1) \times O(V_2) \leftarrow (K^\times \times K^\times)/F^\times$

$\underline{(xy^{-1}, x\bar{y}^{-1})} \leftarrow (x, y)$

$K \hookrightarrow B$

$\alpha \cdot (f_B \times f_B)$.

$B = K \oplus K^\perp$

$V = V_1 \oplus V_2$.

f $\qquad$ $(\chi \times \chi)$.

$$\alpha \cdot \int (f_B \times f_B) \cdot (\chi \times \chi) = \int \theta(\chi \times 1)\big|_{GL_2} \cdot f.$$

$$= \int f \cdot \overline{\theta(\chi)} \cdot \theta(1).$$

$$\alpha \frac{L_\chi(f_B)^2}{\Omega_{CM}} = \frac{\langle f\,\theta(1), \theta(\chi)\rangle}{\Omega_{CM}} = \text{value at } s = 1/2.$$

$\langle f. E(s), \theta(\chi)\rangle$.

$\|$

integral. rep for. $L(s, f_K, \chi)$

value of an $\underline{\text{Eis. series}}$. $E(s)$.

- Harris Kudla: L-value is rational.
- P. (2003). L-values are integral (use Iwasawa Main conj. for imag. quad fields. (Rubin)).
- $\underline{\text{Factorization}}$: (p-adic families)..