K. Prasanna March 13, 2011 (Lec. 2)

Petersson inner product.

$F = \mathbb{Q}$, B indefinite, f on $\Gamma \hookrightarrow SL_2(\mathbb{R})$ (wt 2).

How do we normalize f?

If $B = M_2(\mathbb{Q})$, use q-expansion.

$f \rightsquigarrow$ section of a line bundle

$f \longmapsto 2\pi i\, f(z)\, dz$ $\quad p \setminus h$

$\in H^0(X_B, \Omega^1)$

X_B has a canonical model $/\mathbb{Q}$.

Can pick a multiple of f that is rational over K_f.

Can do better: Fix some prime p, $p \nmid$ level N

X_B has an integral model over $\mathbb{Z}[\frac{1}{N}]$.

$\therefore$ Can normalize f up to p-units in K_f.

Same ideas work over totally real fields.

Petersson inner product: $F = \mathbb{Q}$,

$$\langle f, f \rangle = \frac{1}{\mathrm{vol}(\mathfrak{h}/\Gamma)} \int_{\Gamma \backslash \mathfrak{h}} f(z)\overline{f(z)} \, \frac{dx\,dy}{y^2}$$

If you think adelically, i.e. f as a form on $B^\times(\mathbb{A})$ (or $GL_2(\mathbb{A})$)

$$\langle f, f \rangle = \int_{Z(\mathbb{A}) GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})} f(g)\overline{f(g)} \, d\mu$$

Tamagawa measure.

These definitions generalize to HMF's.

Ω_+, Ω_-.

$$\langle f, f \rangle \sim_{\overline{\mathbb{Q}}^\times} \Omega_+ \Omega_- \qquad (F = \mathbb{Q})$$

~~How are~~ $\langle f_B, f$

F = totally real field $\quad [F:\mathbb{Q}] = d$

$$\Sigma_{\infty,F} = \{v_1, v_2, \ldots, v_d\}$$

B = quat. algebra $/F$.

f a HMF of wt $(2,2,\ldots,2)$

B_{v_i} = quat algebra split at v_i & ramified at v_j, for $j \neq i$.

$X_{v_i} = X_{B_i}$ = a Shimura curve.

Assume for now f transfers to B_{v_i} $\forall i$.

B is split at $\{\tau_1, \ldots, \tau_n\}$, ramified at other infinite places.

$$\{\tau_1, \ldots, \tau_n\} \subseteq \{v_1, \ldots, v_d\}$$

$$X_B \times X_{z_1} \times \cdots \times X_{z_n}$$

B, B' have complementary ramification at ∞

$$(X_B \times X_{B'}) \times X_{M_2(F)}$$

B_1, B_2 two quat algs, same ramification at ∞

$$X_{B_1} \times X_{B_2}$$

Shimura's conjecture.

$$\langle f_B, f_B \rangle \langle f_{B'}, f_{B'} \rangle \sim_{\overline{\mathbb{Q}}^\times} \langle f, f \rangle$$

Proved By Shimura.

$$\langle f_{B_1}, f_{B_1} \rangle \sim_{\overline{\mathbb{Q}}^\times} \langle f_{B_2}, f_{B_2} \rangle$$

Consequences for Period Relations:

Q: How are $\langle f_B, f_B \rangle$ related as B varies?

Conjecture (Shimura): $\exists$ a set of invariants $c_{v_1}, \ldots, c_{v_d} \in \mathbb{C}^\times$, such that

$$\langle f_B, f_B \rangle \sim_{\overline{\mathbb{Q}}^\times} \prod_{B \text{ is split at } v_i} c_{v_i}$$

Michael Harris proved Shimura's conjecture under the following assumption:

(*) $\exists$ at least one finite place v at which π_v is discrete series.

J-L: $f \rightsquigarrow$ an automorphic representation π of $GL_2(\mathbb{A}_F)$

$$\pi = \bigotimes_v \pi_v$$

$$f \text{ transfers to } B^\times \iff \underbrace{\Sigma_B}_{\substack{\text{set of places} \\ \text{where } B \text{ is ramified}}} \subseteq \Sigma(\pi)$$

$$\Sigma(\pi) = \{ v \mid \pi_v \text{ is discrete series} \} \supseteq \Sigma_\infty$$

Harris: theta correspondence for unitary groups.

→ also can remove assumption (*)

Joint work with Ichiro : quaternionic unitary gps.

→ can prove Shim's conjecture without (*).

→ Integrality issues.

Q : Can we say something more precise ?

$F = \mathbb{Q}$, $f \in S_2(\Gamma_0(N))$. $N = N^+ \cdot N^-$

$\operatorname{disc} B = N^-$.

Like to formulate a conjecture:

Need to study

Congruences of Modular forms.

258 D, E   Eg. of congruences between newforms of same level.

$$\begin{cases} \underline{258A} : & + \; + \; 1 \; -5 \; 1 \; -3 \; 0 \; -7 \; \ldots \\ \underline{129A} : & 0 \; + \; -2 \; -2 \; -5 \; 3 \; -3 \; 2 \; \ldots \end{cases}$$

Mod 3 congruence. :

Can we predict such congruences?

Hida, Ribet, Wiles & Taylor-Wiles.

$N = \mathrm{cond}(f)$

- Hida: $\dfrac{\langle f, f\rangle}{\Omega_+ \Omega_-}$ is a measure of congruences;

$\ell \mid \dfrac{\langle f, f\rangle}{\Omega_+ \Omega_-} \iff \exists\, g$ of level $\mid N$, s.t. $f \equiv g \ (\ell)$

$\bar{\rho}_{f,\lambda} \simeq \bar{\rho}_{g,\lambda}$

$\langle f, f\rangle \doteq L(\mathrm{Sym}^2 f, 2)$

$= L(\mathrm{ad}^0 f, 1)$

$(1-\alpha_p p^{-s})(1-\beta_p p^{-s})$

$\shortparallel$

$L_p(f,s)$

$L_p(\mathrm{ad}^0 f, s)$

$= (1 - \alpha_p/\beta_p \cdot p^{-s})$

$\cdot (1 - \beta_p/\alpha_p \cdot p^{-s})$

$\cdot (1 - p^{-s})$.

- Ribet: Level-raising / Level-Lowering.

f level N, $p \nmid N$.

Can you find g of level Np, s.t. $f \equiv g \ (\lambda)$, $\lambda \mid \ell$.

Ribet $\iff$ $\ell \mid L_p(\mathrm{ad}^0 f, 1)$

f level pN; Can you find g of level N, s.t. $f \equiv g \ (\ell)$.

$\bar{\rho}_{f,\lambda} \simeq \bar{\rho}_{g,\lambda}$.

Ribet: If $\bar{\rho}_{f,\lambda}$ is unramified at p, then this is true.

$f \longleftrightarrow$ isogeny class of elliptic curves.

$N =$ sq-free. Irreducible.

$p \mid N$; $\quad \bar{\rho}_{f,\lambda}$ is unramified at p

$\Leftrightarrow \ \ell \mid c_p$ = order of the component gp of the Neron model of E at p.

(E any curve in this isogeny class).

(Tate parametrization).

· Wiles, Taylor-Wiles:

Wiles: η-invariant. (precise measure of congruences).

· $\boxed{\eta\text{-inv} = \dfrac{\langle f, f \rangle}{\Omega_+ \Omega_-}}$

· η-inv $\longleftrightarrow$ order of a Selmer gp.
