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Periods of Modular Forms

Period Relations

V variety / a number field F

ω an algebraic differential form on V, rational over F.

$F \xhookrightarrow{\sigma} \mathbb{C}$

γ a closed top cycle on $V^\sigma(\mathbb{C})$. $\int_\gamma \omega$

Motivating Problem: Tate Conjecture.

$V_1, V_2 / F$

$Gal(\bar{F}/F) \curvearrowright H^*_{et}(V_1), H^*_{\acute{e}t}(V_2)$.

Suppose there is a common (irred). Galois rep in $H^*_{et}(V_1), H^*_{et}(V_2)$.

Tate Conjecture $\cdot \Rightarrow$ a correspondence on $V_1 \times V_2$ that realizes this isomorphism.

$$Z \subseteq V_1 \times V_2$$

$\Rightarrow$ Relations between periods on V_1 and periods on V_2.

- Can we prove such period relations without knowing the Tate Conjecture ?

Two reasons :

(i) Periods occur as transcendental parts of special values of L-functions.

(ii) Hope: gives some ideas on constructing algebraic cycles.

Examples: Langlands Functoriality. .

Jacquet-Langlands Correspondence:

f classical modular form $\in S_2(\Gamma_0(N))$.

f newform. $f = \sum a_n q^n$ $a_1 = 1$.

$\mathbb{Q}(a_n) = K_f$ = number field.

$f \rightsquigarrow$ Galois representation. $\lambda | \ell$

λ any prime of K_f. $\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(K_{f,\lambda})$

<u>Shimura</u> characterized:

(i) $p \nmid N\ell \Rightarrow \rho_{f,\lambda}$ is unramified at p.

(ii) Char. poly of $\rho_{f,\lambda}(\mathrm{Frob}_p)$

$$= T^2 - a_p T + p$$

$X_0(N)$ $J_0(N)$

$H^1_{et}(X_0(N), \mathbb{Q}_\ell)$.

It might happen that f transfers to an indefinite quaternion algebra / $\mathbb{Q}$, in that case $\rho_{f,\lambda}$ can be realized on certain Shimura curves.

$\boxed{nr, tr}$ $\quad x \to x^{\iota}$ (Main involution), $nr(x) = x x^{\iota}$ $tr(x) = x + x^{\iota}$.

B a quat algebra / $\mathbb{Q}$. (central simple algebra / $\mathbb{Q}$ that is 4-dim).

$x^2 = a,\ y^2 = b\ \ xy = -y.x$.

SPLIT.

$$B \otimes \mathbb{Q}_v = \begin{cases} M_2(\mathbb{Q}_v) & : \text{for all but finitely many } v. \\ \text{unique (upto isomorphism) quaternion division algebra} / \mathbb{Q}_v & : \text{a finite set of } v, \text{ of even cardinality.} \end{cases}$$

RAMIFIED.

Suppose B is indefinite: $\boxed{B \otimes \mathbb{R} \overset{i}{=} M_2(\mathbb{R})}$.

Let $\mathcal{O}$ be an order in B. (subring of B that is rank $4 / \mathbb{Z}$).

(eg. $B = M_2(\mathbb{Q})$; (i) $\mathcal{O} = M_2(\mathbb{Z})$.

(ii) $\mathcal{O}_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) / c \equiv 0\ (N) \right\}$)

Let $\Gamma = \{ x \in \mathcal{O} / nr(x) = 1 \}$

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[rr, bend right, hook] & B \otimes \mathbb{R} \arrow[r, "\sim"] & M_2(\mathbb{R}) \\ & & SL_2(\mathbb{R}) \end{tikzcd}$$

Γ is discrete in $SL_2(\mathbb{R})$

$\Gamma \backslash \mathfrak{h}$ (eg: $\mathcal{O}_N$; $\Gamma = \Gamma_0(N)$, $Y_0(N)$).

If $B \neq M_2(\mathbb{Q})$, then $\Gamma \backslash \mathfrak{h}$ is already compact.

Can define modular forms, Hecke operators in usual way. But q-expansions are not available.

$X_B = \Gamma \backslash \mathfrak{h}$ is a complex curve.

<u>Shimura:</u> X_B has a canonical model / $\mathbb{Q}$.

<u>Characterized</u> as follows: $\underset{\text{imag. quad.}}{K} \hookrightarrow B$ $\quad B \otimes \mathbb{R} \overset{i}{\simeq} M_2(\mathbb{R})$

$$K^\times \hookrightarrow GL_2(\mathbb{R})^+$$

$K^\times$ acts on $\mathfrak{h}$, this action has a unique fixed pt τ.

$$\begin{aligned} \mathfrak{h} &\longrightarrow \Gamma \backslash \mathfrak{h} \\ \tau &\longmapsto [\tau] =: P_\tau \end{aligned}$$

Require P_τ be algebraic, further that $Gal(\overline{\mathbb{Q}}/K)$ acts on such P_τ in a prescribed way.

X_B, Modular form g wt 2,

g is an Eigenform for Hecke algebra.

Do we get new systems of Hecke eigenvalues?

Eichler, J-L : No, In fact all systems of Hecke eigenvalues appear on $M_2(\mathbb{Q})$.

J-L: criterion for a system of Hecke eigr on $M_2(\mathbb{Q})$ to appear on B.

$H^1(X_B)$ g on X_B : $\rho_{g,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(K_{g,\lambda})$

$\updownarrow$ $\qquad\qquad\qquad\qquad$ $\wr\!\wr$

$H^1(X_{M_2(\mathbb{Q})})$ f on $X_{M_2(\mathbb{Q})}$: $\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(K_{f,\lambda})$

Tate conj: $\Rightarrow$ cycle on $X_B \times X_{M_2(\mathbb{Q})}$ that realizes this.

Faltings $\Rightarrow$ this is okay.

(i) There is no known canonical construction.

(ii) What if wt $f > 2$. (Scholl: Motives)

(iii) Replace $\mathbb{Q}$ by a totally real field F. (HMF's).

F totally real field. (eg: $F = \mathbb{Q}(\sqrt{d})$, $d>0$, $F = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$)

Hilbert Modular forms: $M_2(F)$

$[F:\mathbb{Q}] = d$, $\Sigma_{F,\infty}$ = set of inf. places of F $= \{v_1, \dots, v_d\}$

$(k_1, \dots, k_d)$ weights.

$(2, 2, \dots, 2)$.

B quat algebra $/F$. X_B : Shimura variety assoc. to B,

$$B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \underbrace{M_2(\mathbb{R})^n}_{\tau_1, \dots, \tau_n} \times \mathbb{H}^{d-n}$$

X_B has dimension: n.

Defined over a reflex field F_B.

$$\begin{tikzcd}[column sep=small]
 & \overline{\mathbb{Q}} \arrow[r, hook] \arrow[d, dash] & \mathbb{C} \\
 & F^{gal} \arrow[dl, dash] \arrow[dr, dash] & \\
F \arrow[dr, dash] & & F_B \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$\mathrm{Hom}(F, \overline{\mathbb{Q}}) = \Sigma_{F,\infty}$

$\sigma_i \longleftrightarrow v_i$

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F_B) = \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \mid \sigma \circ \{\tau_1, \dots, \tau_n\} = \{\tau_1, \dots, \tau_n\}\}$$

eg. (i) ~~If~~ B split at v_1, ram at $v_2, \dots, v_d$: $F_B = F$.

(ii) If B split at v_i, ram at $v_j \neq v_i$: $F_B = \sigma_i F$

Suppose f is a HM newform.

$\rho_{f,\lambda}$: $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow GL_2(K_{f,\lambda})$.

If you had a quat alg B as in eg (i),
then X_B/F, dim 1. $H^1(X_B)$.

<u>General B:</u> X_B / F_B ;

Natural rep in middle dim cohom has dim $= 2^n$.
Can be constructed from $\rho_{f,\lambda}$ by "tensor induction".

$$\boxed{X_B \qquad X_{\tau_1} \times \cdots \times X_{\tau_n}.}$$

$$\rho_B \Big|_{\mathrm{Gal}(\overline{\mathbb{Q}}/F^{gal})} \xrightarrow{\sim} \rho_{\tau_1} \otimes \cdots \otimes \rho_{\tau_n}.$$

ρ_{τ_i} : constructed from a B_{τ_i} (split at τ_i: ram. elsewhere).

(i) $X_B \times X_{z_1} \times X_{z_2} \times \cdots \times X_{z_n}$

(ii) Suppose B, B' have complementary ramification at ∞.

$$(X_B \times X_{B'}) \times X_{M_2(F)}.$$

$$\langle f_B, f_B \rangle \cdot \langle f_{B'}, f_{B'} \rangle \sim_{\overline{\mathbb{Q}}^\times} \langle f, f \rangle$$

(iii) Suppose B, B' have same ramification at ∞.

$$X_B \times X_{B'}$$

$$\langle f_B, f_B \rangle \sim_{\overline{\mathbb{Q}}^\times} \langle f_{B'}, f_{B'} \rangle$$

Shimura's conjecture.

How are $\langle f_B, f_B \rangle$ related as B varies?