p-adic L-functions
&
the Eigencurve

(G. Stevens March 14, 2011
Pollack-Stevens Lec. 4)

$N \geq 1$, $p \nmid N$ prime, $k_0 \geq 0$

$\Gamma = \Gamma_0(N)$, $\Gamma_0 = \Gamma_0(pN) \leq \Gamma$

$\Gamma_0 = \Gamma_0(p) \cap \Gamma$

Motivating Thm: (Hida, Coleman)

Let $f \in M_{k_0+2}(\Gamma_0)$ Hecke eig. form

$f | U_p = \alpha_p f$, $0 \leq \mathrm{ord}_p(\alpha_p) < k_0+1$

Then $\exists B \subseteq \mathcal{W} = \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$

$k_0 \in B$, B an affinoid disk

& $\mathbb{f} \in A(B)[[q]]$, $\mathbb{f} = \sum_{n \geq 0} \alpha_n q^n$

s.t. ① $\mathbb{f}_k := \sum \alpha_n(k) q^n$, $k \in (B \cap \mathbb{Z}^{\geq 0})$

then $\mathbb{f}_k$ is $\in M_{k+2}(\Gamma_0)$

+ ② $\mathbb{f}_{k_0} = f$

Exples: ⓐ $\mathbb{G}_k := -\frac{1}{2}\zeta_p(-1-k) + \sum_{n \geq 1} \sigma_{k+1}^*(n) q^n$

$$\sigma_{k+1}^*(n) = \sum_{\substack{d \mid n \\ p \nmid d}} d^{k+1}$$

ⓑ $f = q \prod_{n \geq 1} (1-q^n)^2 (1-q^{11n})^2 \in S_2(\Gamma_0(11))$

$\Delta = q \prod_{n \geq 1} (1-q^n)^{24} \in S_{12}(\Gamma(1))$

$\Delta_\alpha = \Delta(z) - \beta \Delta(pz)$

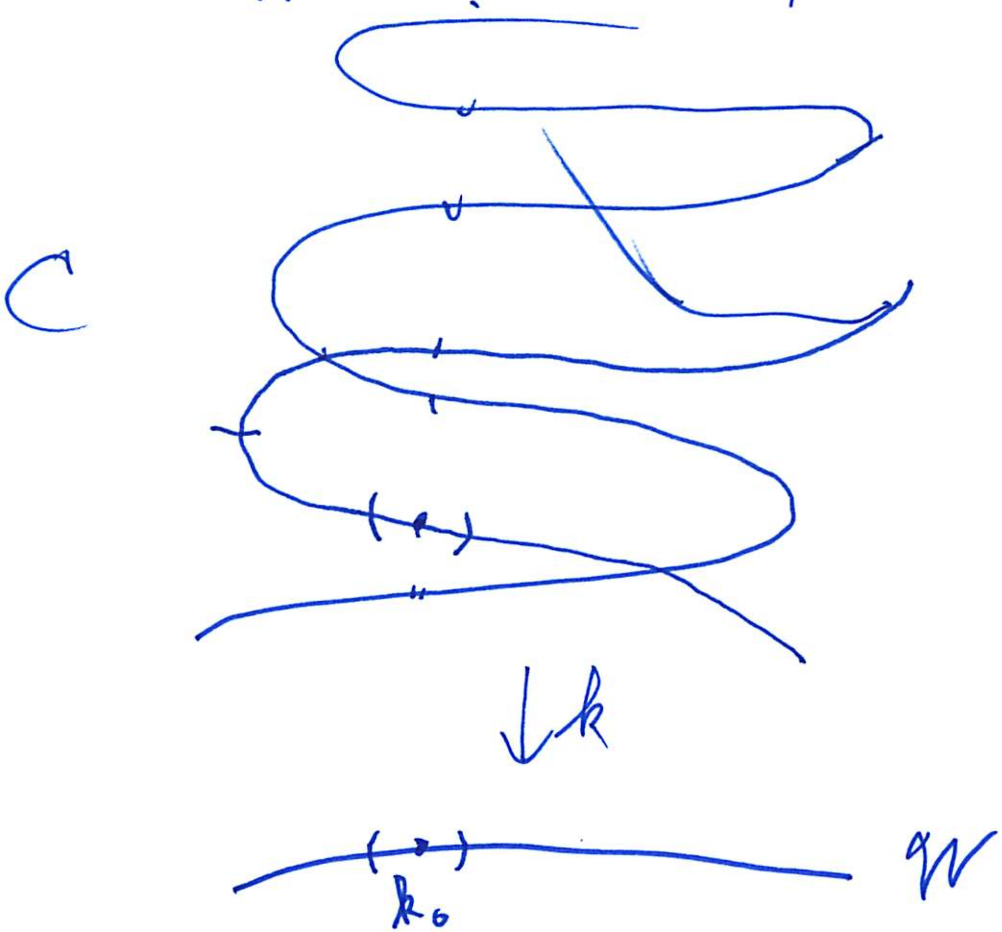
The eigencurve

Thm (Coleman-Mazur-Buzzard)

$\exists\ C$ = a p-adic rigid analytic curve

$k \downarrow$ locally finite map

$\mathcal{W}$ = "weight space"



s.t

(a) $\exists$ 1-1 correspondence (for any fixed $k_0 \in \mathcal{W}$)

$$\left\{ \begin{array}{c} x \in C(K) \\ \text{s.t. } k(x) = k_0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{o.c. eigenforms} \\ f_x \in M^{\dagger}_{k_0+2}(\Gamma_0, K) \\ \alpha_p(f_x) \neq 0 \end{array} \right\} \quad K/\mathbb{Q}_p$$

& (b) $\exists$ rigid analytic fns

$$\alpha_n : C \longrightarrow \mathbb{C}_p \quad \text{s.t.}$$

$$\forall x \in C(K), \quad f_x = \sum_{n \geq 0} \alpha_n(x) q^n$$

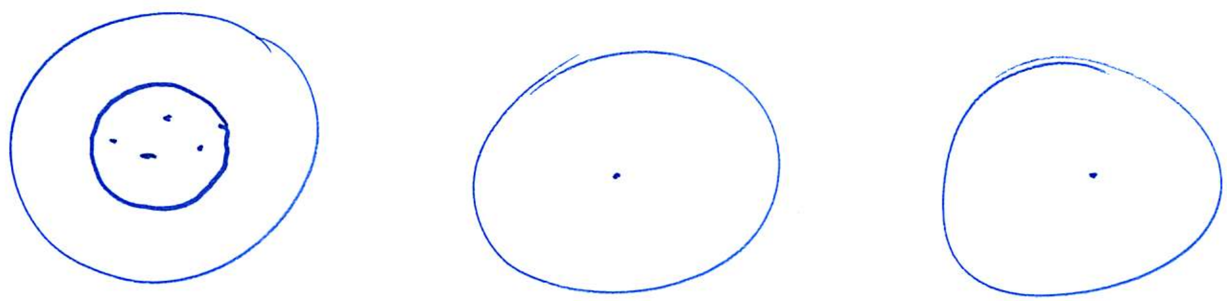
Moreover, C is smooth & unramified$/\mathcal{W}$ at every classical point x ~~of noncritical slope~~ (Bellaiche)

Thm (Amice - Velu) If $0 \le \mathrm{slope}(f) < k_0 + 1$

then $\exists!$ $L_p(f, \cdot)$ satisfying

1. Interpolation.
2. $L_p(f)$ is tempered of order $\mathrm{slope}(f)$

$$L_p(f, \cdot) : \mathcal{W} \longrightarrow \mathbb{C}_p$$



$$f \in S_{k_0+2}(\Gamma_0)$$

$$\omega_f = 2\pi i \, f(z)\,(z\underline{X} + \underline{Y})^{k_0}\,dz$$

$$\in \Omega^1(\mathcal{H}) \otimes \underbrace{\mathrm{Symm}^{k_0}(\mathbb{C})}_{L_{k_0}}$$

$$\Psi_f(r \to s) = \int_r^s \omega_f \in L_k$$

$$\Lambda(\Psi_f) \in L_k$$

Modular Symbols

$\varphi \in \mathrm{Symb}_{\Gamma_0}(M)$ $\quad M$ a Γ_0-module

$$\doteq \mathrm{Hom}_{\Gamma_0}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)$$

Thm: (Ash, S.) If $\#\mathrm{tor}(\Gamma_0)$ acts invertibly on M then

$$H^1_c(\Gamma_0, M) \cong \mathrm{Symb}_{\Gamma_0}(M)$$

$\varphi \in H^1_c(\Gamma_0, M)$

$$\varphi(r \to s) := \varphi(\{s\} - \{r\}) \in M.$$

$$\Lambda(\varphi) = \varphi(\infty \to 0) \quad \text{Universal } L\text{-value.}$$

Locally analytic Distributions

$\mathcal{A}(\mathbb{Z}_p) =$ locally analytic functions on $\mathbb{Z}_p$

$$\mathcal{D}(\mathbb{Z}_p) = \mathcal{A}(\mathbb{Z}_p)^*$$

$$\begin{array}{ccc} \mathcal{D}(\mathbb{Z}_p) & \xrightarrow{\text{res}} & \mathcal{D}(\mathbb{Z}_p^\times) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mu & & \mu|_{\mathbb{Z}_p^\times} \end{array}$$

Defn: $L_p(\mu, s) = \int_{\mathbb{Z}_p^\times} t^{s-1} \, d\mu(t) \in \mathbb{C}_p$

rigid analytic for $s \in \mathcal{W}$

Theorem: (Amice)

$$\begin{aligned} \mathcal{D}(\mathbb{Z}_p^\times) &\xrightarrow{\sim} A(\mathcal{W}) \\ \mu &\longmapsto L_p(\mu, \cdot) \end{aligned}$$

Γ_0 acts on $\mathcal{A}(\mathbb{Z}_p)$, depending on $k \in \mathcal{W}$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0$

$$(\gamma f)(z) = (a+cz)^k f(z\gamma)$$

$\Gamma_0 \circlearrowright \mathcal{A}_k(\mathbb{Z}_p)$, $\Gamma_0 \circlearrowright \mathcal{D}_k(\mathbb{Z}_p)$

Study:

$$H^1_c(\Gamma_0, \mathcal{D}_k(\mathbb{Z}_p))$$

$k_0 \geq 0 \Rightarrow$

$$H^1_c(\Gamma_0, \mathcal{D}_{k_0}) \longrightarrow H^1_c(\Gamma_0, L_{k_0})$$

If $0 \leq h < k_0 + 1$ the

$$H^1_c(\mathcal{D}_{k_0})^{(\leq h)} \cong H^1_c(\cancel{\ } L_{k_0})^{(\leq h)}$$

$$\mathcal{P}_{k_0} \subseteq \mathcal{A}_{k_0}$$

polynomials

$$L_{k_0} = \mathcal{P}_{k_0}^{*} \longleftarrow \mathcal{D}_{k_0}$$

$$f \in \cancel{\Phi} S_{k_0+2}(\ )^{(<k_0+1)}$$

$$\varphi_f^{\pm} \in H_c^1(\Gamma_0, \cancel{\mathcal{D}} L_{k_0})^{(<k_0+1)}$$

$$\uparrow \qquad \cong$$

$$\Phi_f^{\pm} \in H_c^1(\Gamma_0, \mathcal{D}_{k_0})^{(<(k_0+1))}$$

$$\mu_f = \Lambda(\Phi_f) \in \mathcal{D}_{k_0}$$

$$L_p(f, s) := L_p(\mu_f, s)$$