$$\mathrm{Hom}_{\Gamma}(\Delta_0, V) =: \mathrm{Symb}_{\Gamma}(V) \longrightarrow H^1(\Gamma, V)$$

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$$\varphi \longmapsto (\gamma \mapsto \varphi(\gamma\infty - \infty))$$

$$\mathbb{A}_k = \left\{ \sum a_n z^n \;\middle|\; |a_n| \to 0 \quad a_n \in \mathbb{Q}_p \right\}$$

$$\mathbb{D}_k = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{A}, \mathbb{Q}_p) \qquad \mathbb{D}_k \longrightarrow V_k(\mathbb{Q}_p)$$

<u>Thm</u>: (Stevens)

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)^{(<k+1)} \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0}(V_k)^{(<k+1)}$$

$$\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)^{(<k+1)} \cong \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k)^{(<k+1)}$$

Idea: $k = 0$ slope $= 0$

Fix φ a classical eigensymbol

w/ U_p eigenvalue λ, $\operatorname{ord}_p(\lambda) = 0$.

Let Φ be any lifting of φ.

U_p is a compact operator

So its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ with valuation $\to \infty$.

Imagine $\Phi = \Phi_1 + \Phi_2 + \cdots + \Phi_n + \cdots$

with Φ_i having eigenvalue λ_i

Assume further that $\mathrm{ord}_p(\lambda_i) > 0$ if $i > 1$.

Apply $\frac{U_p}{\lambda}$ again and again.

$$\underbrace{\frac{1}{\lambda^N}\Phi|U_p^N}_{\text{still lifts } \varphi} = \frac{\lambda_1^N}{\lambda^N}\Phi_1 + \frac{\lambda_2^N}{\lambda^N}\Phi_2 + \cdots + \frac{\lambda_n^N}{\lambda^N}\Phi_n + \cdots \to 0$$

$$\rho^*\left(\frac{1}{\lambda^N}\Phi|U_p^N\right) = \frac{1}{\lambda^N}\rho^*(\Phi)|U_p^N = \frac{1}{\lambda^N}\varphi|U_p^N$$
$$= \varphi$$

$$\left\{\frac{1}{\lambda^N}\Phi|U_p^N\right\} \longrightarrow \Phi_1$$

Make this more rigorous.

1. Need to justify that Φ exists.

2. $\left\{ \frac{\Phi \mid U_p^N}{\lambda^N} \right\}$ prove is Cauchy converging to say $\tilde{\Phi}$

3. $\tilde{\Phi}$ is an eigensymbol lifting φ.

   ($\tilde{\Phi}$ is indep. of lift)

③ $\tilde{\Phi}$ lifts φ.

This is true since each $\frac{\Phi | \mathcal{V}_p^N}{\lambda^N}$ does.

$\tilde{\Phi} | \mathcal{V}_p = \lambda \tilde{\Phi}$?

$$\tilde{\Phi} | \mathcal{V}_p = \left(\lim_{N \to \infty} \frac{1}{\lambda^N} \Phi | \mathcal{V}_p^N \right) | \mathcal{V}_p$$

$$= \lim_{N \to \infty} \frac{1}{\lambda^N} \Phi | \mathcal{V}_p^{N+1}$$

$$= \lambda \left(\lim_{N \to \infty} \frac{1}{\lambda^{N+1}} \Phi | \mathcal{V}_p^{N+1} \right)$$

$$= \lambda \tilde{\Phi}$$

(2) $\left\{ \dfrac{\Phi | U_p^N}{\lambda^N} \right\}$ is Cauchy.

$N \leq M$

$$\frac{\Phi | U_p^N}{\lambda^N} - \frac{\Phi | U_p^M}{\lambda^M}$$

$$= \frac{1}{\lambda^N} \Big(\underbrace{\Phi - \frac{\Phi | U_p^{M-N}}{\lambda^{M-N}}}_{} \Big) | U_p^N$$

in the kernel of specialization.
(since both specialize to φ)

Claim: $\Psi \in \ker(\rho^*) \Longrightarrow \Psi | \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ is
$\|\Psi\| = 1$ divisible by p.

((Claim $\Longrightarrow$ Cauchy)

Subclaim:

$$\mu \in \mathbb{D},\ \mu(1) = 0,\ \|\mu\| = 1 \implies \mu\left(\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}\right) \text{ is divisible by } p.$$

(Subclaim $\Rightarrow$ claim).

Pf of subclaim:

$$\left(\mu \,\middle|\, \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}\right)(z^j) = \mu\left(\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \cdot z^j\right)$$

$$= \mu\left((a+pz)^j\right)$$

$$= \mu\left(a^j + \binom{j}{1} a^{j-1} pz + \ldots + (pz)^j\right)$$

$$= \mu\left(p\left(\underline{\qquad\qquad}\right)\right)$$

(4) Uniqueness of $\tilde{\Phi}$

Picke 2 lift Φ and Φ' (of φ).

$$\Phi - \Phi' \in \ker(\rho^*)$$

By claim $(\Phi - \Phi') \mid \mathcal{V}_p^N \longrightarrow 0.$

Approximating distributions

$$\mu \in \mathbb{D} \longleftrightarrow \{\mu(z^j)\}_{j=0}^{\infty}$$

Fix $M, N \gg 0$. Consider the first N-moments of μ each mod p^M.

Unfortunately, this data is __not__ stable under the action of $\Sigma_0(p)$.

(i.e. this data for μ does not determine the same data for $\mu|\gamma$).

Indeed, ~~μ~~ $\mu(z^j) = 0 \quad 0 \le j \le N$

$\not\Rightarrow$ $\mu|\gamma$ has the same property.

Let $\mu_4 \in \mathbb{D}$

$$\mu_4(z^j) = \begin{cases} 1 & j = 4 \\ 0 & j \neq 4 \end{cases} .$$

$$\gamma = \begin{pmatrix} 1 & 0 \\ -p & 1 \end{pmatrix}$$

$$(\mu_4|\gamma)(1) = \mu_4(\gamma . 1) = \mu_4(1) = 0$$

$$(\mu_4|\gamma)(z) = \mu_4(\gamma . z) = \mu_4\left(\frac{z}{1-pz}\right)$$

$$= \mu_4\left(z . (1 + pz + p^2z^2 + \ldots)\right) = p^3$$

$$(\mu_4|\gamma)(z^2) = \mu_4\left(\left(\frac{z}{1-pz}\right)^2\right)$$

$$= \mu_4\left(z^2(1 + pz + p^2z^2 \ldots)^2\right) = 3p^2$$

$$(\mu_4|\gamma)(z^3) = \ldots = 3p$$

$\mathbb{D}^\circ$ = unit ball of $\mathbb{D}$. $\quad (\mu(z^j) \in \mathbb{Z}_p)$

$$\mathrm{Fil}^M(\mathbb{D}^\circ) = \left\{ \mu \in \mathbb{D}^\circ \;\middle|\; \mu(z^j) \text{ is div. by } p^{M-j} \right\}$$

$0 \le j \le M$.

<u>Fact</u>: $\mathrm{Fil}^M(\mathbb{D}^\circ)$ is stable under $\Sigma_0(p)$.

<u>Defn</u>: $\mathcal{F}(M) = \mathbb{D}^\circ / \mathrm{Fil}^M(\mathbb{D}^\circ)$ ← finite approx module.

$\Sigma_0(p) \circlearrowright \mathcal{F}(M) \simeq \mathbb{Z}/p^M \times \mathbb{Z}/p^{M-1} \times \cdots \times \mathbb{Z}/p$

$$\mathbb{D}^\circ = \varprojlim \mathcal{F}(M)$$

Reprove the comparison thm (M. Greenberg)

Modify the filtration:

$$\widetilde{Fil}^M(\mathbb{D}^0) = Fil^M(\mathbb{D}^0) \cap \ker(\rho)$$

$$\tilde{\mathcal{F}}(M) = \frac{\mathbb{D}^0}{\widetilde{Fil}^M(\mathbb{D}^0)} \simeq \mathbb{Z}_p \times \mathbb{Z}/p^{M-1} \times \cdots \times \mathbb{Z}/p$$

$$\begin{array}{ccc}
 & & \mathrm{Symb}_{\Gamma_0}(\mathbb{D}^0) \\
 & & \downarrow \\
\varphi_N & \in & \mathrm{Symb}_{\Gamma_0}(\mathcal{F}(M)) \\
\downarrow & & \downarrow \\
\varphi & \in & \mathrm{Symb}_{\Gamma_0}(\mathbb{Z}_p)
\end{array}$$

Build φ_M taking values in $\tilde{\mathcal{F}}(M)$

$$\varphi_M \longrightarrow \varphi_{M-1}, \quad \varphi_M | U_p = \lambda \varphi_M .$$

Assume we have such a φ_M.

(Build φ_{M+1})

$$\varphi_M \in \mathrm{Hom}_{\Gamma_0}(\Delta_0, \hat{\mathcal{F}}(M))$$

Pick $\psi_{M+1} : \Delta_0 \longrightarrow \hat{\mathcal{F}}(M+1)$ lifting φ_M.

$$\psi_{M+1} \in \mathrm{Maps}(\Delta_0, \hat{\mathcal{F}}(M+1))$$

Magic: $\varphi_{M+1} := \psi_{M+1} \Big| \dfrac{U_p}{\lambda}$

φ_{M+1} is additive, Γ_0-inv, indep of lift and satisfies $\varphi_{M+1} | U_p = \lambda \varphi_{M+1}$.

Why the magic?

$$\varphi_{M+1}(D+D') - \varphi_{M+1}(D) - \varphi_{M+1}(D')$$

$$= \frac{1}{\lambda} \sum_{a=0}^{p-1} \underbrace{\left(\psi_{M+1}(\gamma_a D + \gamma_a D') - \psi_{M+1}(\gamma_a D) - \psi_{M+1}(\gamma_a D')\right)}_{\widetilde{\mathrm{Fil}}^{M}(\mathbb{D}^{0})} \Big| \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$$

$\gamma_a = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$

Check that $\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ kills this