Pollack Lec 2, 3-12-2011

p-adic families

$p \nmid N$

Let $S_k(\Gamma_0(Np))^{ord}$ denote the space of p-ordinary forms.

(eigenvalue U_p is a p-adic unit)

Fact: $\dim(S_k(\Gamma_0(Np))^{ord})$ only depends on k modulo $p-1$.

ex: $\underbrace{S_2(\Gamma_0(15))}_{\text{1-dim'l}}$ $p=3$

$q - q^2 - q^3 - q^4 + q^5 + \cdots$

Fact $\Rightarrow$ $\dim(S_k(\Gamma_0(151))^{ord}) = 1$

for all even k.

Let f_k denote the unique normalized ordinary form.

Moreover,

$f_k \equiv f_{k'} \ (3^n)$ when $k \equiv k' \ (3^{n-1})$

"Hida theory"

Hida interpolates the space $S_k(\Gamma_0(N_p))^{ord}$ as k varies.

Clearly, this can't be true for all forms,

$\dim(S_k(\Gamma)) \to \infty$ as $k \to \infty$.

Coleman replaced $M_k(\Gamma_0)$ w/ $M_k^\dagger(\Gamma_0)$

overconvergent modular forms.
∞-dim'l space.

$\Gamma_0 = \Gamma_0(Np)$

- $M_k(\Gamma_0) \hookrightarrow M_k^\dagger(\Gamma_0)$
- $f \in M_k^\dagger(\Gamma_0)$ with "small slope", then $f \in M_k(\Gamma_0)$
- $M_k^\dagger(\Gamma_0)$ interpolates ~~are~~ p-adically.

Modular Symbols analogue

Recall $\mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k) \longleftrightarrow M_{k+2}(\Gamma_0)$

Replace ~~w~~ V_k w/ D_k some ∞-dim'l space of distributions.

$$D_k \twoheadrightarrow V_k$$

$$\mathrm{Hom}_{\Gamma_0}(\Delta_0, D_k) \longrightarrow \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k)$$

Distributions

Let $\mathbb{A} = \{$ conv power series over $\mathbb{Q}_p$ on the closed unit disc of $\mathbb{C}_p \}$

$$= \left\{ \sum_{n=0}^{\infty} a_n z^n \;\middle|\; a_n \in \mathbb{Q}_p,\ |a_n|_p \to 0 \right\}$$

$\mathbb{A}$ is a Banach space

$$\left\| \sum a_n z^n \right\| = \max_n |a_n|$$

$\mathbb{D} = \mathrm{Hom}_{\text{cont}}(\mathbb{A}, \mathbb{Q}_p)$ again a Banach space.

<u>Notation</u>: $\mu \in \mathbb{D}$ $\quad f \in \mathbb{A}$

$$\mu(f) =: \int f \, d\mu$$

Moments

- the span of the monomials $\{z^j\}_{j=0}^{\infty}$ is dense in A.

$\Rightarrow$ $\mu \in D$ is uniquely determined by the sequence $\{\mu(z^j)\}_{j=0}^{\infty}$

$\longrightarrow$ $$D \hookrightarrow \prod_{j=0}^{\infty} \mathbb{Q}_p$$

$$\mu \longmapsto (\mu(z^j))$$

In fact, the image = bounded seqns in $\mathbb{Q}_p$.

Take any bnded seqn $\{\alpha_n\}$.

Want $\mu(z^j) = \alpha_j$

$$\mu\left(\sum a_n z^n\right) := \sum a_n \alpha_n$$

$\mathbb{D} \longleftrightarrow$ bnded seqn in $\mathbb{Q}_p$

$\mathbb{A} \longleftrightarrow$ seqn in $\mathbb{Q}_p$ tending to 0.

Matrix actions

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}_p) \ : \ p \nmid a \ , \ p \mid c \right\}$$

Fix $k \geq 0$, $\gamma \in \Sigma_0(p)$, $f \in \mathbb{A}$.

$$(\gamma \cdot_k f)(z) := (a+cz)^k \cdot f\left(\frac{b+dz}{a+cz}\right)$$

$\mu \in \mathbb{D}$

$$(\mu|_k \gamma)(f) := \mu(\gamma \cdot_k f)$$

Write $\mathbb{A}_k$ and $\mathbb{D}_k$.

We consider $\mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D}_k)$ to be the space of overconvergent modular symbols (OMS) of weight k.

Specialization

$$\rho_k : \mathbb{D}_k \longrightarrow V_k(\mathbb{Q}_p) = \mathrm{Sym}^k(\mathbb{Q}_p^2)$$

$$\mu \longmapsto \int (Y - zX)^k \, d\mu$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \mu(z^j) X^j Y^{k-j}$$

this is $\Sigma_0(p)$-equivariant.

Ex: $k=0$

$$\rho_0 : \mathbb{D} \longrightarrow \mathbb{Q}_p$$
$$\mu \longmapsto \mu(1)$$

$$(\mu|\gamma)(1) = \mu(\gamma \cdot 1) = \mu(1) = \mu(1)|\gamma$$

$$\rightsquigarrow \rho_k^* : \mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D}) \longrightarrow \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\mathbb{Q}_p))$$

Specialization.

Slopes of modular forms

$f \in S_k(\Gamma_0)$ eigenform $\quad f|U_p = \lambda f$

slope of $f = \mathrm{ord}_p(\lambda)$ $\qquad \mathrm{ord}_p(p) = 1$

p-adic valuation

Fact: slope $f \leq k-1$

Reason: ① p-new $\Rightarrow a_p = \lambda = \pm p^{\frac{k}{2}-1}$ ✓

② p-old $\Rightarrow \exists\, g$ on $\Gamma_0(N)$ s.t.

$f \in \mathrm{span}(g(z), g(pz))$ $\circlearrowright U_p$ $\qquad \mathrm{Char}(U_p) = X^2 - a_p X + p^{k-1}$

$\Rightarrow$ valuation of the roots $\leq k-1$ ✓

$h \in \mathbb{R}$. $M^{(<h)}$ = subspace of M where

U_p acts with slope $< h$.

Thm (Stevens)

$$\mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D}_k)^{(<k+1)} \xrightarrow{\sim} \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k)^{(<k+1)}$$

(Analogue of Coleman's "slope small" $\Rightarrow$ classical thm)

Let $f \in S_{k+2}(\Gamma_0)$ eigenform

$$\rightsquigarrow \Psi_f^{\pm} \in \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\mathbb{C}))$$

$$\rightarrow \varphi_f^{\pm} = \Psi_f^{\pm}/\Omega_f^{\pm} \in \mathrm{Hom}_{\Gamma_0}(\Delta_0, V_k(\bar{\mathbb{Q}}_p))$$

$$\varphi_f = \varphi_f^{+} + \varphi_f^{-}$$

Assume slope of f $< k+1$.

By control thm $\exists!$ Φ_f Hecke-eigensymbol lifting φ_f.

<u>Thm (Stevens)</u>

$\Phi_f(0 - \infty)$ = p-adic L-function of f.

p-adic L-functions

$L_p(f) =$ gadget "knows" $\frac{L(f, \chi, 1)}{\Omega_f^{\pm}} \in \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

cond p^n

$$\chi \xrightarrow{L_p(f)} c \cdot \frac{L(f, \chi, 1)}{\Omega_f^{\pm}}$$

$L_p(f) =$ distribution.

i.e. dual to some nice space of functions.

$$L_p(f) \in \mathbb{D} = \mathrm{Hom}(\mathbb{A}, \mathbb{Q}_p)$$

$$\chi \notin \mathbb{A}.$$

$\mathcal{A}$ = locally analytic fns on $\mathbb{Z}_p$

$$\mathbb{A} \xhookrightarrow{\text{dense}} \mathcal{A}$$

$$\mathcal{D} = \mathrm{Hom}(\mathcal{A}, \mathbb{Q}_p) \qquad \mathcal{D} \hookrightarrow \mathbb{D}$$

$h \in \mathbb{R}$

<u>Fact</u>: $\mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathcal{D})^{(<h)} \xrightarrow{\sim} \mathrm{Hom}_{\Gamma_0}(\Delta_0, \mathbb{D})^{(<h)}$