R. Pollack AWS March 2011

Lecture 1, 3-12-2011

Overconvergent Modular Symbols

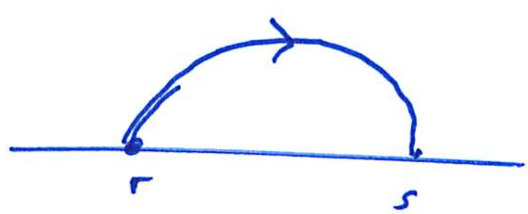
Period Integrals

R. Pollack I

March 12, 2011

Let f be a cuspform of weight 2 on $\Gamma_0(N)$.

$$\boxed{2\pi i \int_r^s f(z)\,dz}$$



$r, s \in \mathbb{P}^1(\mathbb{Q})$

$$2\pi i \int_{i\infty}^{0} f(z)\,dz = L(f,1)$$

$$L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

$$f = \sum_{n \geq 1} a_n q^n$$

$$q = e^{2\pi i z}$$

Twists

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$

$$L(f,\chi,s) = \sum_{n\geq 1} \frac{a_n \chi(n)}{n^s}$$

$$L(f,\chi,1) = c \cdot \sum_{a \,(\mathrm{mod}\, N)} \chi(a)\, 2\pi i \int_{\infty}^{a/N} f(z)\,dz$$

Bogus argument

$$2\pi i \int_{i\infty}^{0} f(z)\,dz = 2\pi i \int_{i\infty}^{0} \sum a_n e^{2\pi i n z}\,dz$$

$$= \sum \frac{a_n}{n} e^{2\pi i n z} \Big|_{i\infty}^{0} = \sum \frac{a_n}{n} = L(f,1)$$

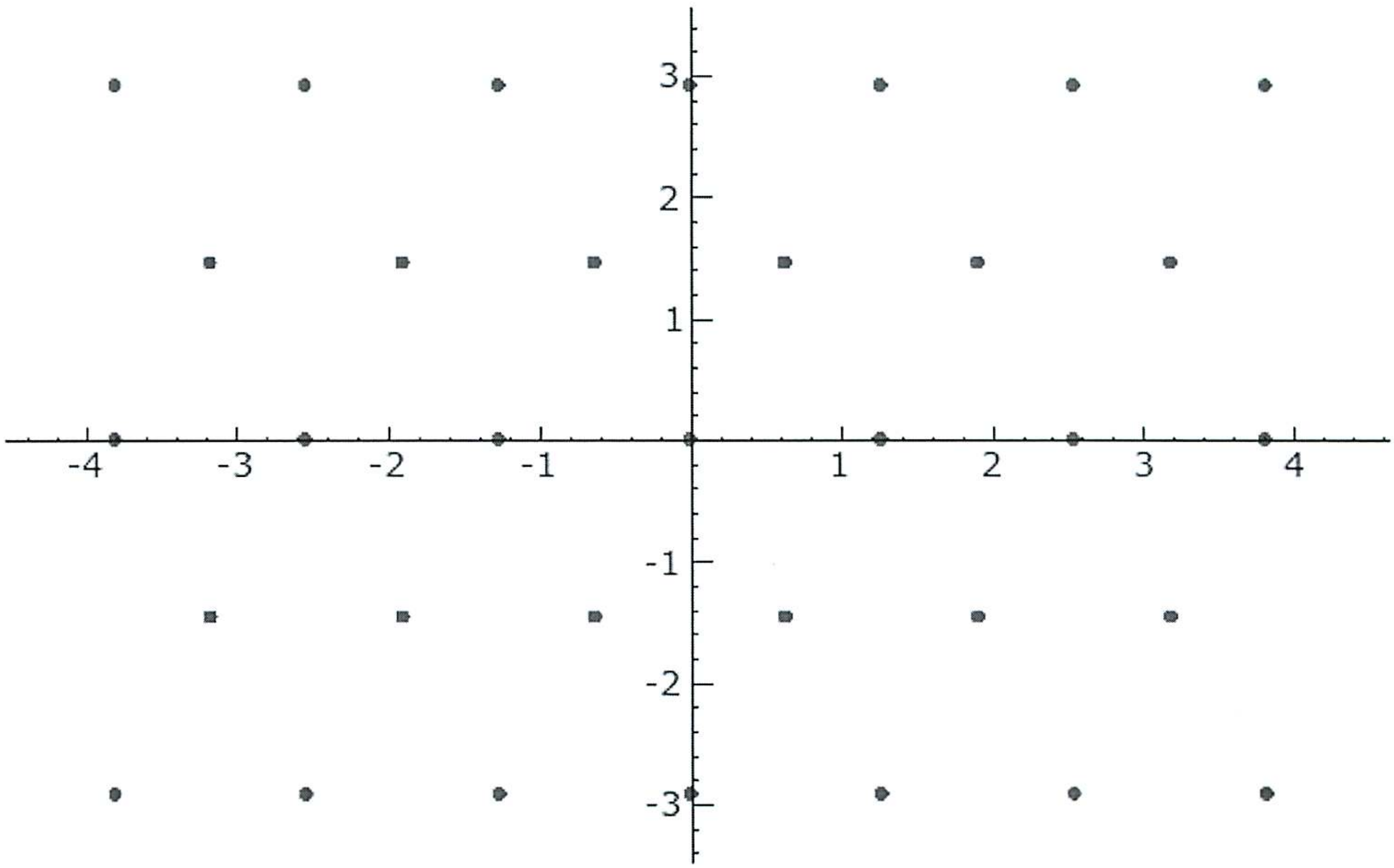
Numerical experiment

$f(z) \in S_2(\Gamma_0(11))$

compute $2\pi i \int_r^s f(z)\,dz$

for bunch of r's and s's

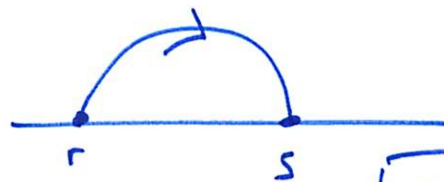
3



Modular Symbols

$$\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) = \text{degree } 0 \text{ divisors on } \mathbb{P}^1(\mathbb{Q})$$

$$\{s\} - \{r\} \longleftrightarrow$$



$$\boxed{f \in S_2(\Gamma_0(N))}$$

$$\Psi_f : \Delta_0 \longrightarrow \mathbb{C}$$

$$\{s\} - \{r\} \longmapsto 2\pi i \int_r^s f(z)\, dz \qquad \text{(extend linearly)}$$

$$\rightsquigarrow \Psi_f \in \mathrm{Hom}(\Delta_0, \mathbb{C})$$

Easy fact: $$\int_r^s f(z)\, dz = \int_{\gamma r}^{\gamma s} f(z)\, dz$$

$$\gamma \in \Gamma_0(N) \qquad \gamma z = \frac{az+b}{cz+d} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow \psi_f(D) = \psi_f(\gamma \cdot D)$$

$D \in \Delta_0$

$$\rightsquigarrow \psi_f \in \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$$

$$\rightsquigarrow S_2(\Gamma_0(N)) \hookrightarrow \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$$

$$f \longmapsto \psi_f$$

$\exists$ involution ι on $\mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$

$$(\iota(\varphi))(D) = \varphi\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} D\right)$$

$$\{s\} - \{r\} \longmapsto \{-s\} - \{-r\}$$

$$\varphi = \varphi^+ + \varphi^-$$

$\underline{\Gamma_0(11)}$: $f \rightsquigarrow \psi_f^+$ takes values in $\mathbb{Z}\cdot\Omega^+$

$\rightsquigarrow \psi_f^-$ " " " $\mathbb{Z}\cdot\Omega^-$

$\Omega^+, \Omega^- \in \mathbb{C}$.

Any thing else here?

Easier way to build symbols.

$$\Delta = \mathrm{Div}(\mathbb{P}^1(\mathbb{Q}))$$

$$\mathrm{Hom}_{\Gamma_0(N)}(\Delta, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C})$$

$\varphi \in$

$=$ fcn on $\mathbb{P}^1(\mathbb{Q})$ constant on orbits of $\Gamma_0(N)$

$$\mathbb{P}^1(\mathbb{Q})/\Gamma_0(11) = \{\infty\} \cup \{0\}.$$

$\uparrow$ $\frac{a}{11c}$ $\qquad$ $\uparrow$ $\frac{a}{c}$ $\quad 11 \nmid c$

$\Rightarrow \mathrm{Hom}_{\Gamma_0(11)}(\Delta, \mathbb{C})$ is 2-dim'l.

$$\varphi_0(x) = \begin{cases} 1 & x \sim \infty \\ 0 & x \sim 0 \end{cases} \qquad \varphi_1(x) = \begin{cases} 0 & x \sim \infty \\ 1 & x \sim 0 \end{cases}$$

$$\varphi_0|_{\Delta_0} = -\varphi_1|_{\Delta_0}$$

Get 1 new modular symbol.

$\Rightarrow \mathrm{Hom}_{\Gamma_0(11)}(\Delta_0, \mathbb{C})$ is at least 3-dim'l.

Any more?

$\Gamma = \Gamma_0(N)$ What info determines a modular symbol?

- Δ_0 generated $\{s\} - \{r\}$

- collection of $\{s\} - \{r\}$ is generated by divisors of the form $\{\frac{b}{d}\} - \{\frac{a}{c}\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

(unimodular path)

Ex: $\{\frac{11}{4}\} - \{\frac{0}{1}\} = \left(\{\frac{11}{4}\} - \{\frac{3}{1}\}\right) + \left(\{\frac{3}{1}\} - \{\frac{-2}{-1}\}\right)$

$$+ \left(\{\tfrac{2}{1}\} - \{\tfrac{1}{0}\}\right) + \left(\{\tfrac{1}{0}\} - \{\tfrac{0}{1}\}\right)$$

Given $\underset{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}{\underset{\shortparallel}{\alpha}} \in SL_2(\mathbb{Z})$, $[\alpha] = \{\frac{b}{d}\} - \{\frac{a}{c}\}$

- φ is determined by its values on $[\alpha]$ $\forall \alpha \in SL_2(\mathbb{Z})$.

- Γ-inv of $\varphi \Rightarrow \varphi([\alpha])$ only depends on α in $\Gamma_0(N) \backslash SL_2(\mathbb{Z})$

$$\Gamma_0(N) \backslash SL_2(\mathbb{Z}) \simeq \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$$

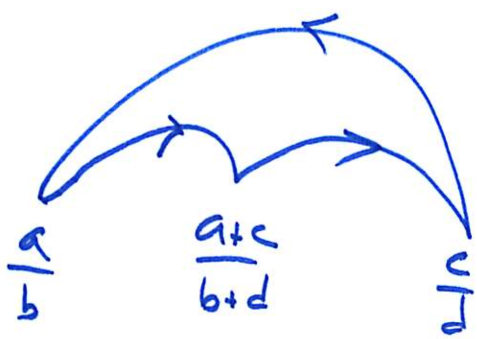
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [c : d]$$

$N = 11$: Mod symbols is at most 12 dim'l.

· Note $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = -\left[\begin{pmatrix} -b & a \\ -d & c \end{pmatrix}\right]$

$$\varphi([c:d]) = -\varphi([-d:c])$$

·



$\longrightarrow$ 3-term relation

"Manin relations"

Solve relations $\Longrightarrow$ Mod symbols $/\Gamma_0(11)$

are 3-dim'l.

Higher weight?

Need non-trivial coefficients

(right)

$V = \mathbb{Z}[\Gamma]$ - module

$\gamma \in \Gamma$

$\varphi \in \mathrm{Hom}(\Delta_0, V)$

right action of Γ

$$(\varphi|\gamma)(D) := \varphi(\gamma D)|\gamma$$

Mod symbols with values in V and level Γ

$$= \mathrm{Hom}_\Gamma(\Delta_0, V)$$

$$- \varphi|\gamma = \varphi \Rightarrow \varphi(\gamma D) = \varphi(D)|\gamma^{-1}$$

Take $V = V_k = \mathrm{Sym}^k(\mathbb{C}^2)$

hom. polys in X, Y of degree k.

$$(P|\gamma)(X,Y) = P\left((X,Y)\gamma^*\right), \qquad \gamma^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$S_{k+2}(\Gamma) \longrightarrow \mathrm{Hom}_\Gamma(\Delta_0, V_k)$$

$$f \longmapsto \underbrace{\left(\{s\}-\{r\} \longmapsto 2\pi i \int_r^s (zX+Y)^k f(z)\,dz\right)}_{\varphi_f}.$$

- $\Psi_f(\gamma D) = \Psi_f(D)|\gamma^{-1}$

- $\Psi_f(0-\infty) = \sum c_i X^i Y^{k-i}$

$$c_i \longleftrightarrow L(f, i+1)$$

<u>Hecke action</u>

$$\mathrm{Hom}_\Gamma(\Delta_0, V_k) \circlearrowleft T_n = \Gamma\begin{pmatrix}1&0\\0&n\end{pmatrix}\Gamma$$

e.g. $\Gamma = \Gamma_0(N)$, $\ell \nmid N$.

$$(\varphi|T_\ell) = \sum_{a=0}^{\ell-1} \varphi|\begin{pmatrix}1&a\\0&\ell\end{pmatrix} + \varphi|\begin{pmatrix}\ell&0\\0&1\end{pmatrix}$$

* $f \rightsquigarrow \Psi_f$ preserves Hecke.

Eichler - Shimura $\exists$ an isom of Hecke-modules.

$$M_{k+2}(\Gamma) \oplus S_{k+2}(\Gamma) \simeq \mathrm{Hom}_{\Gamma}(\Delta_0, V_k)$$

Rmks

1. $M_{k+2}(\Gamma) \longleftrightarrow$ plus part
2. $S_{k+2}(\Gamma) \longleftrightarrow$ minus part.
3. eisenstein series $\longleftrightarrow$ "boundary symbols" on Δ
4. RHS is completely ~~alg~~ algebraic and completly computable.

$V_k = \mathrm{Sym}^k(\mathbb{C}^2)$ replace w/ $\mathrm{Sym}^k(\mathbb{Q}^2)$, $\mathrm{Sym}^k(\mathbb{Z}^2)$