S. Dasgupta March 15, 2011 (Lecture 5) ①

Siegel's cocycle

(for $P \in \mathcal{P}_d$)

$$\psi_T(\gamma)(P,v) = \frac{(d+1)!}{(2\pi i)^{d+2}} \int_T^{\gamma T} P(z,1) E_{d+2,v}(z)\,dz$$

$\psi_T \in Z^1(\Gamma, M)$:

$$\underbrace{\int_T^{ABT} P\cdot E_{d+2,v}}_{\psi_T(AB)(P,v)} = \underbrace{\int_T^{AT} P\cdot E_{d+2,v}}_{\psi_T(A)(P,v)} + \underbrace{\int_{AT}^{ABT} P\cdot E_{d+2,v}}$$

$$\int_{AT}^{ABT} P\cdot E_{d+2,v} = \int_T^{BT} (A^T P) E_{d+2,A^{-1}v} = (A\psi_T(B))(P,v).$$

We did a heuristic calculation

$$\psi_T(\gamma)(1,v) \text{``}=\text{''} \sum_{m,n\in\mathbb{Z}}{}' \frac{e(mv_1+nv_2)\cdot(\gamma\tau-\tau)}{(m\cdot(\gamma\tau)+n)(m\tau+n)}$$

plug in $\tau = r/s \in \mathbb{Q}$

$\sigma_1 = \begin{pmatrix} r \\ s \end{pmatrix}$ $\sigma_2 = \gamma\begin{pmatrix} r \\ s \end{pmatrix}$, $\sigma = (\sigma_1, \sigma_2)$

$$\Phi(\gamma)(1,v) \text{``}:=\text{''} \frac{1}{(2\pi i)^2} \sum_{z=(m,n)\in\mathbb{Z}^2}{}' \frac{\det(\sigma)\, e(\langle z,v\rangle)}{\langle z,\sigma_1\rangle\langle z,\sigma_2\rangle}$$

Problems: den $=0$?

convergence?

Fix problems generalize to $n \geq 2$

For $i = 1, \ldots, m$ let Q_i = linear form in n variables whose coeffs are lin. indep. over $\mathbb{Q}$

$$Q = \prod_{i=1}^{m} Q_i \qquad \Gamma \curvearrowright \mathcal{L} = \{\text{such } Q\}$$

Let $\Gamma = SL_n(\mathbb{Z})$.

Fix $A_1, \ldots, A_n \in \Gamma$

(in this example w/ $n=2$ $A_1 = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$
$A_2 = \gamma \cdot A_1$)

For $z \in \mathbb{Z}^n$, $z \neq 0$

let σ_i = first column of A_i s.t. $\langle z, \sigma_i \rangle \neq 0$

$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Define

$$\boxed{\begin{array}{c}\Phi(A_1, \ldots, A_n)(1, Q, v) \\ \parallel \\ \dfrac{1}{(2\pi i)^n} \lim\limits_{t \to \infty} \sum\limits_{\substack{z \in \mathbb{Z}^n \\ |Q(z)| < t}}{}' \dfrac{\det(\sigma) \cdot e(\langle z, v \rangle)}{\langle z, \sigma_1 \rangle \cdots \langle z, \sigma_n \rangle}\end{array}}$$

$\Phi(A_1, \ldots, A_n)(P, Q, v)$ def'd similarly.

$$M = \{ f : \mathcal{P}_{\mathbb{Q}} \times \mathcal{L} \times \mathbb{Q}^n / \mathbb{Z}^n \longrightarrow \mathbb{Q} \}$$

linear in $\mathcal{P}_{\mathbb{Q}}$-variable

dist. prop. in $\mathbb{Q}^n/\mathbb{Z}^n$ variable.

Thm(s) (Sczech)

1. $\Phi(A_1, \ldots, A_n) \in M$, i.e.
   $\Phi(A_1, \ldots, A_n)(P, Q, v) \in \mathbb{Q}$!
   explicit formula involving Dedekind Sums.

2. $\Phi \in Z^{n-1}(\Gamma, M)$
   (homogeneous cocycle)

3. Siegel's formula holds:

F = tot real field w/ $[F:\mathbb{Q}] = n$.

$$\zeta_{K/F,R}(\sigma_{\mathfrak{a}}, 1-r) = \Phi(\underbrace{A_1, \ldots, A_n}_{\text{"1 \& basis of } E(f)\text{"}})(\underset{\substack{\parallel \\ \text{Norm wrt dual basis of } \langle w_1, \ldots, w_n\rangle}}{P^{r-1}}, Q, v)$$

$K = K_f$

~~Norm wrt dual basis of $\langle w_1, w_2\rangle$~~

1 + 3 $\Longrightarrow$ Klingen–Siegel that

$$\zeta_{K/F,R}(\sigma_{\mathfrak{a}}, 1-r) \in \mathbb{Q}$$

Thm (Charollois - D) There is an ℓ-smoothed version $\Phi_\ell \in Z^{n-1}(\Gamma_0(\ell), M_\ell)$ such that the integrality thm holds.

$$\Gamma_0(\ell) = \begin{pmatrix} * & & \\ 0 & & \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix} \pmod{\ell}$$

Cor $\zeta_{K/F,R,T}(\sigma_{\mathfrak{a}}, 1-r) \in \mathbb{Z}[\frac{1}{\ell}]$

$\in \mathbb{Z}$ if $r=1$, $\ell \geq n+2$.

[Thm of Deligne-Ribet/ Cassou-Noguès/ Barsky

Measures

Fix p prime. $v \in \mathbb{Q}^n/\mathbb{Z}^n$.

$pv \equiv v \mod \mathbb{Z}^n$ $\quad$ $(\mathrm{denom}(v) \mid p-1)$

(will suppress "$\mathbb{Q}$")

Define $\mu(\underbrace{A_1, \ldots, A_n}_{A})(v)$, a

$\mathbb{Z}[\frac{1}{\ell}]$-valued measure on $\mathbb{Z}_p^n$:

$$\mu(A)(v)(v + a + p^r \mathbb{Z}_p^n) = \underbrace{\Phi_\ell(A)\left(1, \frac{v+a}{p^r}\right)}_{\in \mathbb{Z}[\frac{1}{\ell}]}$$

$a \in \mathbb{Z}^n$

Thm $\quad \Phi_\ell(A)(P, v) = \int_{\mathbb{Z}_p^n} P(x)\, d\mu(A)(v)(x).$

Given TR field of degree n, F,
s.t. p is inert in F

~~#~~ Suppose $p \equiv 1 \pmod{f}$

Have associated data

$$P, v, A_1, \ldots, A_n, Q \ldots.$$

define for $s \in W = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$

$$\zeta_p(\sigma_{\mathfrak{a}}, s) = \int_{\mathbb{Z}_p^n - p\mathbb{Z}_p^n =: \underline{X}} P(x)^{-s} \, d\mu(A)(v)(x)$$

Pvs thm $\Rightarrow$ $\zeta_p(\sigma_{\mathfrak{a}}, 1-r)$

$$\zeta_p(\sigma_{\mathfrak{a}}, 1-r) = \zeta_{K/F, S, T}(\sigma_{\mathfrak{a}}, 1-r), \quad S = R \cup \{p\}$$

for $r \in \mathbb{Z}^{\geq 1}$

new construction of p-adic zeta functions.

Gross' Conj:

$$\zeta_p'(\sigma_{\mathfrak{a}}, 0) = -\log_p \mathrm{Norm}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(u_T^{\sigma_{\mathfrak{a}}})$$

$$\| $$

$$-\int_{\mathbb{X}} \log_p \mathrm{Norm}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(x_1 w_1 + \cdots + x_n w_n)\, d\mu(A)(v)(x)$$

$\boxed{\text{Conj}}$ $\log_p(u_T^{\sigma_{\mathfrak{a}}}) =$

$$\int_{\mathbb{X}} \log_p(x_1 w_1 + \cdots + x_n w_n)\, d\mu(A)(v)(x)$$