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(Dasgupta–Greenberg Lec 3)

Shintani Zeta Functions & Stark Units II

0 Recall

$a \in M_{n\times d}(\mathbb{C})$, $\mathrm{Re}(a_i^j) > 0$, $x \in \mathbb{R}^d_{\geq 0}$, $x \neq 0$.

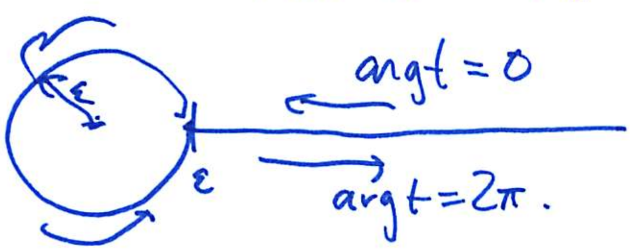
$$\zeta(a,x,s) = \sum_{k\in\mathbb{Z}^d_{\geq 0}} N(a(x+k))^{-s} \qquad Nv = v_1\cdots v_n.$$

$$\mathrm{Re}(s) > \frac{d}{n}.$$

$$\zeta(a,x,s) = \sum_{i=1}^{n} z_i(a,x,s), \text{ where}$$

$$z_i(a,x,s) = c_n(s)\overbrace{\int_{C(\infty,\varepsilon)} \frac{du}{u}\, u^{ns} \int_{C(1,\varepsilon)^{n-1}} G(uy|_{y_i=1}) \prod_{r\neq i} y_r^{s}\frac{dy_r}{y_r}}^{J_a(s)}.$$

$$c_n(s) = \frac{1}{\Gamma(s)^n\,(e^{2\pi i n s}-1)(e^{2\pi i s}-1)^{n-1}}, \quad G(t) = \prod_{j=1}^{n} \frac{e^{t a^j (1-x_j)}}{e^{t a^j}-1}.$$



(ε small enough).

1 Values of $\zeta(a,x,s)$ at nonpos. integers

Residue calculus gives:

$$z_i(a,x,1-m) = \frac{(-1)^{n(m-1)}}{n} \frac{B_{i,m}(\mathbb{1}-x)}{m!} \qquad \mathbb{1}=(1,\dots,1)$$

$m \geq 1$.

$$\frac{B_{i,m}(\mathbb{1}-x)}{(m!)^n} = \operatorname{coeff}\left(G(uy|_{y_i=1}), (u^n y_1 \cdots \hat{y}_i \cdots y_r)^{m-1}\right)$$

Eg: value at $s=0$.

$$I_\varepsilon(0) = \int_{C(\infty,\varepsilon)} \frac{du}{u} \int_{C(1,\varepsilon)^{n-1}} G(uy|_{y_i=1}) \frac{1}{y_1 \cdots \hat{y}_i \cdots y_r} dy_1 \cdots \hat{dy_i} \cdots dy_r .$$

$$= (2\pi i)^{n-1} \int_{C(\infty,\varepsilon)} \frac{du}{u} \prod_{j=1}^{d} \frac{e^{ua_i^j(1-x_j)}}{e^{ua_i^j}-1}$$

$$\prod_{j=1}^{d} = \prod_{j=1}^{d} \frac{1}{ua_i^j} \underbrace{\frac{(ua_i^j)\, e^{ua_i^j(1-x_j)}}{e^{ua_i^j}-1}}$$

$$= \prod_{j=1}^{d} \frac{1}{ua_i^j} \sum_{k_j=0}^{\infty} \frac{B_{k_j}(1-x_j)}{k_j!} (ua_i^j)^{k_j}.$$

$$z_i(a,x,0) = \frac{(-1)^d}{n} \sum_{\substack{l_1+\cdots+l_d=d \\ \text{ordered} \\ (l_1,\dots,l_d)}} \prod_{j=1}^{d} \frac{B_{l_j}(x_j)}{l_j!} (a_i^j)^{l_j-1}.$$

$$z_i(a,x,0) \in \mathbb{Q}(\{x_j\},\{a_i^j\}).$$

<u>Thm</u> (Klingen-Siegel) $\zeta_{K/F}(\sigma_{\sigma_1}, 1-m) \in \mathbb{Q}$.

<u>Pf</u> (Shintani) $a^1, \dots, a^d \in F_{>0}$.

$$\zeta(a,x,0) = \sum_{l} \underbrace{\mathrm{Tr}_{F/\mathbb{Q}}\left(\prod_{j=1}^{d} \frac{B_{l_j}(x_j)}{l_j!}(a^j)^{l_j-1}\right)}_{\text{Dedekind Sums.}} \in \mathbb{Q}.$$

$(x_j \in \mathbb{Q})$.

(2) Shintani zeta fns for non-TR fields.

F number field $\quad v_i : F \hookrightarrow \mathbb{R}.\ i=1,\dots,r_1$
$\quad v_i : F \hookrightarrow \mathbb{C} \quad i=r_1+1,\dots n.$

$\mathfrak{f} \subset \mathcal{O}_F$, $K=K_{\mathfrak{f}}$

$\exists\ \mathcal{C}$. cones s.t.

$$F_{>0} = \coprod_{\varepsilon \in E(\mathfrak{f})} \coprod_{c \in \mathcal{C}} \varepsilon c.$$

$$c = \sum_{j=1}^{d(c)} \mathbb{Q}_{>0}\, a_c^j \qquad a^j \in F_{>0}.$$

Problem: No guarantee $\mathrm{Re}(a_{c,i}^j) > 0$.

Fix: At cost of refining $\mathcal{C}$. $\exists\, u_{c,i} \in \mathbb{C}$, $c \in \mathcal{C}$, $i = 1, \dots, n$ s.t.

1. $|u_{c,i}| = 1 \quad \forall i$
2. $v_i = \overline{v_{i'}} \implies u_{c,i} = \overline{u_{c,i'}}$
3. $\mathrm{Re}(u_{c,i}\, a_{c,i}^j) > 0$.

$$u_c = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & u_{c,r+1} & \\ & & & & \ddots \\ & & & & & u_{c,n} \end{pmatrix}.$$

$$(2) \implies N(u_c a_c (x+k)) = N(a_c(x+k)).$$

$$\implies \zeta_{K/F,1,S}(\sigma_{\mathfrak{a}}, s) = \sum_{c \in \mathcal{C}} \sum_{\substack{z \in (1+\sigma_{\mathfrak{a}}^{-1}\mathfrak{f}) \cap C \\ z = \sum x_j a^j \\ x_j \in (0,1]}} \zeta(u_c a_c, x, s).$$

Define:

$$Z_i(\sigma, \mathcal{C}, \{u_c\}, s) = \sum_{c \in \mathcal{C}} \sum_{z \in \dots} Z_i(u_c a_c, x, s)$$

$$N\mathfrak{a}^s \cdot \zeta_{K/F}(\sigma_{\mathfrak{a}}, \Delta) = \sum_{i=1}^{n} z_i(\mathfrak{a}, \mathcal{C}, \{u_c\}, \Delta).$$

Eg F imag quad.

$$\zeta'_{K/F}(\sigma, 0) = -\log|u^\sigma|_w \quad \forall \sigma \in Gal(K/F)$$
$$= -(\underbrace{\log^w(u^\sigma)}_{z_1'(0)} + \underbrace{\log^w(\overline{u^\sigma})}_{z_2'(0)}).$$

Thm (Shintani) $z_1'(-, 0) = \log$ (CM-value of Siegel unit)

(We'll see why indep. of choices).

$n > 2$ $z_i'(\mathfrak{a}, \mathcal{C}, \{u_c\}, 0)$ not indep. of choice.

3 Dependence on choices:

Thm. 1 $z_i(\mathfrak{a}, \mathcal{C}, \{u_c\}, 0)$ is well defined.

2 If v_i is real $z_i'(\mathfrak{a}, \mathcal{C}, \{u_c\}, 0)$ is well defined.

3 Suppose v_i is cplx. $N(c,x) = \text{denom}(\zeta(u_c a_c, x, 0))$

~~Then~~

$$\varphi_i(a_c, x) := z_i'(u_c a_c, x, 0) - \underbrace{\zeta(u_c a_c, x, 0)}\log(u_{c,i})$$

~~is indep. of~~

$\varphi_i(a_c, x)$ mod $\dfrac{2\pi i \mathbb{Z}}{N(c,x)}$ is ~~well defined~~ indep. of $\{u_c\}$

4 $$\Phi_i(\sigma, \mathcal{C}) := \sum_{c \in \mathcal{C}} \sum_{\substack{\mathfrak{z} \in \cdots \\ \cdots}} \varphi_i(a_c, x).$$

$$N(\sigma, \mathcal{C}) = \text{lcm}(\{N(c,x)\}).$$

$\Phi_i(\sigma, \mathcal{C})$ mod $\dfrac{2\pi i}{N(\sigma, \mathcal{C})}\mathbb{Z}$ is ~~well~~ indep of $\{u_c\}$.

$\propto$ $$\sum_{i=1}^{n} \Phi_i(\sigma, \mathcal{C}) = \zeta'_{K/F,S}(\tau_\sigma, 0).$$

Worthwhile project: "T-smoothing" this whole discussion, relate to

Stark-Tate $\zeta'_{K/F,S,T}(0) = -\log|u_T^\sigma|_w$.

(u_T unique if exists).

Smoothing & Shintani:

* Pi Casson-Nouges const^{n} of p-adic zeta function of totally real field.
* Dasgupta's refinement conjecture.

From now on I'll ignore all N's.

Thm v_i cplx, $\mathcal{C}, \mathcal{C}'$. $\exists\, \varepsilon \in E(f)$ s.t.

$$\Phi_i(\sigma, \mathcal{C}_e') = \Phi_i(\sigma, \mathcal{C}_e) + \log \varepsilon_i \pmod{2\pi i \mathbb{Z}}$$

Pf. • $\Phi_i(\sigma, \mathcal{C}_e)$ is invariant under subdivision of $\mathcal{C}_e$. (Sczech.)

• It suffices to consider: $\forall c \in \mathcal{C}_e\ \exists\, \varepsilon_c \in E(f)$ s.t.

$$\mathcal{C}_e' = \{\varepsilon_c c : c \in \mathcal{C}_e\}.$$

Then $a_{\varepsilon_c c} = \delta_c a_c \quad \delta_c = \mathrm{diag}(\varepsilon_{c,1}, \ldots, \varepsilon_{c,n})$.

can take $u_{\varepsilon_c c} = u_c \delta_c^{-1}$.

$$\varphi(a_{\varepsilon_c c}, x) - \underbrace{\zeta(u_{\varepsilon_c} c a_c, x, 0)}\log u_{c_{\mathfrak{F}}, i}\,\varepsilon_{c,i}^{-1}$$

$$\equiv \varphi(a_c, x) - \overbrace{\zeta(u_c a_c, x, 0)}\log u_{c_{\mathfrak{F}}, i}$$

$$\Rightarrow \varphi(a_{\varepsilon_c} c, x) \equiv \varphi(a_c, x) + \log \varepsilon_{c,i}^{\zeta(u_c a_c, x, 0)}$$

Now sum over x, c, ... $\square$.

④ <u>Complex cubic ~~fields~~ fields</u> (Ren-Sczech).

F cplx cubic σ_1 real emb, $\overline{\sigma_2} = \sigma_3$

$E(f) = \langle \eta \rangle$.

<u>Want</u>: $\vartheta \bmod 2\pi i \mathbb{Z}$ some comb. of

$\Phi_1(\sigma, \tau_c)$ & $\Phi_2(\sigma, \tau_c)$ s.t.

① $\vartheta \bmod 2\pi i \mathbb{Z}$ is well defined.

② $-(\vartheta + \overline{\vartheta}) = \zeta'_{K/F, S}(\sigma_\sigma, 0)$.

<u>Conj</u>. $\vartheta = \log(\text{stark unit})$.

$$\vartheta = \phi_1(\ell_c, \sigma) \frac{\log \eta_2}{\log \eta_1} - \phi_2(\ell_c, \sigma).$$

$$\cancel{\vartheta + \bar{\vartheta} = \phi_1 \frac{\log \eta_2 + \log \eta_3}{\log \eta_1} - \phi_2 - \phi_3.}$$

Exercise: Check 1 & 2 :)