Hi we are

Shintani zeta functions & stark units I.

M. greenberg

March 13, 2011

(Dasgupta-Greenberg Lec. 2)

Cases known.

TR_∞ $F = \mathbb{Q}$. example.

ATR F imag quadratic (elliptic units)

TR_p. $F = \mathbb{Q}$ Stickelberger's Thm.
Gross' refinement
Other totally real base field
(under hypotheses)
Darmon - Dasgupta - Pollack.

Only F s.t. Stark is known $\forall K/F$
is $F = \mathbb{Q}$ or F = imag quad.

1 Motivation

F imag quad, K/F abel.

$S = \infty_F \cup$ places of F ram. in K.

$v = \infty_F$.

$w \mid v$

Stark: $\zeta'_{K/F,S}(\sigma, 0) = \frac{1}{e} \log |u^\sigma|_w$.

$v_1, v_2 : F \hookrightarrow \mathbb{C} \quad (\bar{v}_1 = v_2)$.

$w_1, w_2 : K \hookrightarrow \mathbb{C}. \quad (\bar{w}_1 = w_2)$

$$\zeta'_{K/F,S}(\sigma, 0) = \frac{1}{e} \log\left(|w_1(u^\sigma)| + \log|w_2(u_\sigma)|\right)$$

CM theory: $w_i(u^\sigma) =$ CM value of an ell. unit.

Idea.

$[F:\mathbb{Q}]=2$ $\zeta_{K/F,S}(\sigma,s) = z_1(s) + z_2(s)$.

- $z_i(s)$ not well defined.
- $z_i'(0)$ is mostly well defined.

* Decomp arises is in Shintani's proof of rationality of ζ-values of totally real fields at nonpositive integers (1976).

2 Shintani Zeta functions

$a \in M_{n\times d}(\mathbb{C})$, $\mathrm{Re}(a_i^j) > 0$ $\forall i,j$.

$x \in \mathbb{R}^d_{\geq 0}$, $x \neq 0$.

__Def__. $\zeta(a,x,s) = \sum_{k\in\mathbb{Z}^d_{\geq 0}} N(a(x+k))^{-s}$

$\mathrm{Re}(s) > \frac{d}{n}$.

$N\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = c_1 \cdots c_n$.

<u>Rk</u>. $n=1$ case : Hurwitz zeta functions.

<u>Thm</u>. $\zeta(a, x, s)$ admit mero ctn to $\mathbb{C}$.

(proof later)

<u>First</u>: Connection with partial zetas.

<u>Eg</u>. $F/\mathbb{Q}$ real quadratic, $x \longmapsto x_i$: embs.

$\mathfrak{f} \subset \mathcal{O}_F$ st $E(\mathfrak{f}) = \langle \varepsilon \rangle$, $0 < \varepsilon_1 < 1$.

$K = K_\mathfrak{f}$ ray class field

$\mathfrak{a} \subset \mathcal{O}_F$ $(\mathfrak{a}, S_{\min}) = 1$. $(\mathfrak{a}, \mathfrak{f}) = 1$.

$$\zeta_{K/F, S}(\sigma_\mathfrak{a}, s) = \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_F \ \mathfrak{b} \sim_\mathfrak{f} \mathfrak{a}. \\ (\mathfrak{b}, S) = 1}} N\mathfrak{b}^{-s}.$$

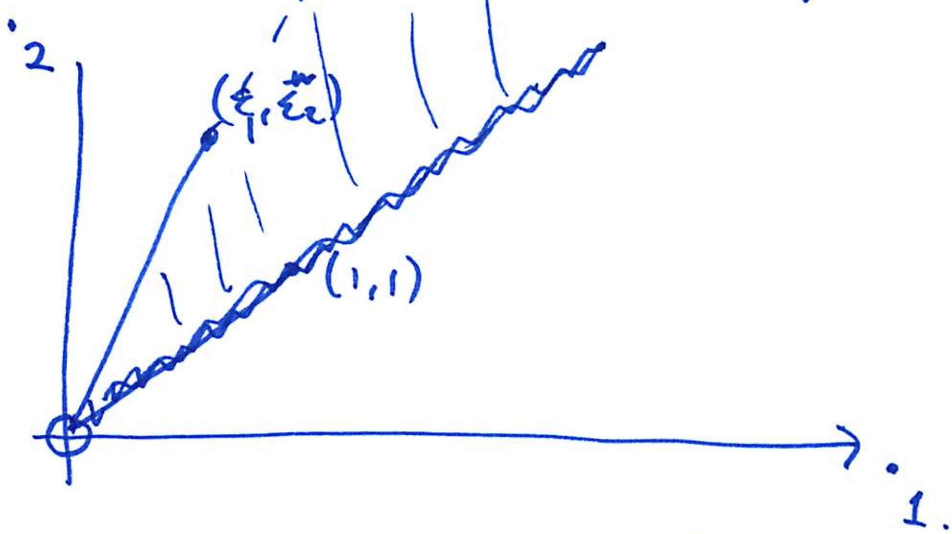
$$= N\mathfrak{a}^{-s} \sum_{z \in (1 + \mathfrak{a}^{-1}\mathfrak{f})_{>0} / E(\mathfrak{f}).} Nz^{-s}$$

<u>Exercise</u>: (i) $(1 + \mathfrak{a}^{-1}\mathfrak{f})_{>0} / E(\mathfrak{f}) \xrightarrow{\sim} \{\mathfrak{b} \subset \mathcal{O}_F : (\mathfrak{b}, S), \mathfrak{b} \sim_\mathfrak{f} \mathfrak{a}\}$

$x \longmapsto x\mathfrak{a}$.

② $F_{>0} = \bigsqcup_{n\in\mathbb{Z}} \varepsilon^n c[1,\varepsilon)$

$$c[1,\varepsilon) = \mathbb{Q}_{>0}\cdot 1 + \mathbb{Q}_{\geq 0}\,\varepsilon .$$



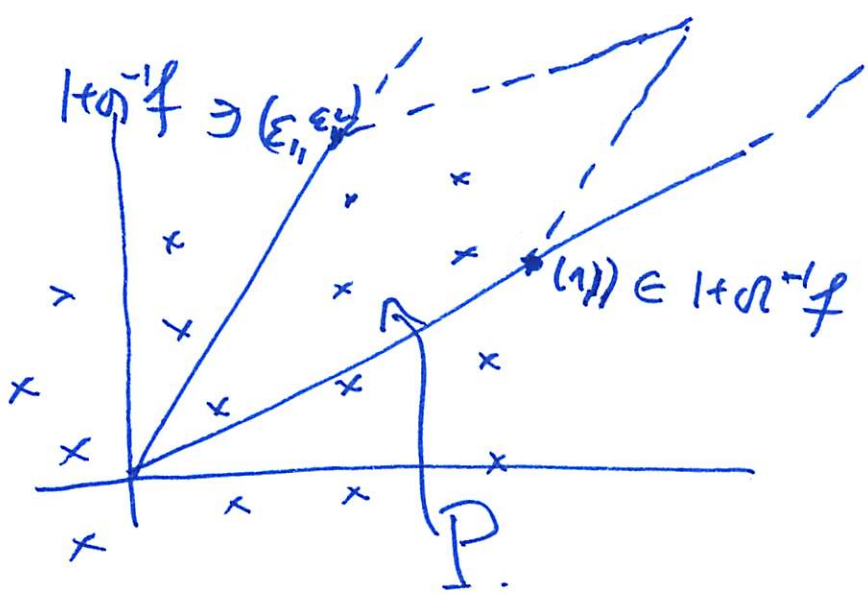
By ② $(1+\mathfrak{a}^{-1}\mathfrak{f})_{>0} \cap c(1,\varepsilon) \xrightarrow{\cong} (1+\mathfrak{a}^{-1}\mathfrak{f})_{>0}/E(\mathfrak{f})$

$$\therefore\ \zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s) = N\mathfrak{a}^{-s} \sum_{z\in(1+\mathfrak{a}^{-1}\mathfrak{f})\cap c[1,\varepsilon)} Nz^{-s}$$

Asymmetric bd. conditions inconvenient

fix : $c[1,\varepsilon) = c(1) \cup c(1,\varepsilon)$.

$$\zeta_{K/F,S}(\sigma_{\mathfrak{a}}, s) = N\mathfrak{a}^{-s}\Big(\sum_{(1+\mathfrak{a}^{-1}\mathfrak{f})\cap c(1)} + \sum_{(1+\mathfrak{a}^{-1}\mathfrak{f})\cap c(1,\varepsilon)}\Big).$$



$$(1+\mathfrak{a}^{-1}\mathfrak{f}) \cap c(1,\varepsilon) = \coprod_{\underbrace{z \in (1+\mathfrak{a}^{-1}\mathfrak{f}) \cap P}_{\text{finite}}} \left(z + \mathbb{Z}_{\geq 0} 1 + \mathbb{Z}_{\geq 0} \varepsilon \right).$$

$$\sum_{(1+\mathfrak{a}^{-1}\mathfrak{f}) \cap c(1,\varepsilon)} Nz^{-s} = \sum_{\substack{z \in (1+\mathfrak{a}^{-1}\mathfrak{f}) \cap c(1,\varepsilon) \\ z = x_1 + x_2\varepsilon,\ x_i \in (0,1] \cap \mathbb{Q}}} \ \sum_{k \in \mathbb{Z}_{\geq 0}^2} N\big((x_1+k_1) + (x_2+k_2)\varepsilon\big)^{-s}.$$

$$= \sum_{z} \sum_{k} N\left(\underbrace{\begin{pmatrix} 1 & \varepsilon_1 \\ 1 & \varepsilon_2 \end{pmatrix}}_{\in M_{2\times 2}(\mathbb{R}_{>0})} \begin{pmatrix} x_1+k_1 \\ x_2+k_2 \end{pmatrix} \right)^{-s}$$

<u>Thm</u> (Shintani) F totally real field, $\exists$ finite set $\mathcal{C}$ of cones in $F_{>0}$ s.t.

$$F_{>0} = \coprod_{\varepsilon \in E(\mathfrak{f})} \coprod_{c \in \mathcal{C}} \varepsilon c.$$

<u>Cor.</u> $N\sigma^s \zeta_{K/F,S}(\sigma_{\sigma}, s)$ is a finite sum of Shintani zeta functions.

3 Analytic continuation & Shintani's decomp

<u>Euler:</u> $\displaystyle\int_0^\infty e^{-nt} t^s \frac{dt}{t} = \Gamma(s) n^{-s}$

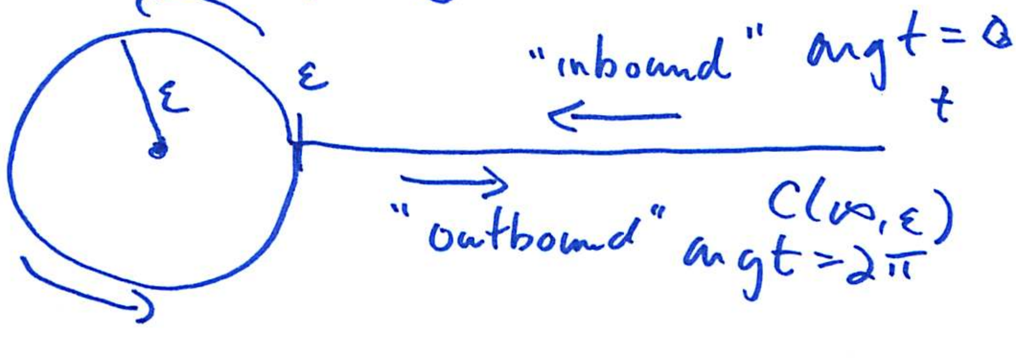
n-dim'l version:

$$\Gamma(s)^n \zeta(a,x,s) = \int_{(0,\infty)^n} e^{-tax} \overbrace{\sum_{k \in \mathbb{Z}^d_{\geq 0}} e^{-tak}}^{\text{prod. of } d \text{ geom series}} t^s \frac{dt}{t}$$

where $t = (t_1, \cdots, t_n)$, $t^s \frac{dt}{t}$ $t_1^s \cdots t_n^s \frac{dt_1 \cdots dt_n}{t_1 \cdots t_n}$.

$$= \int_{(0,\infty)^n} G(t) t^s \frac{dt}{t}, \text{ where}$$

$$G(t) = \prod_{j=1}^d \frac{e^{ta^j(1-x_j)}}{e^{ta^j} - 1}.$$



$$I_\varepsilon = \int_{C(\infty,\varepsilon)^n} \prod_{j=1}^{d} \frac{e^{ta^j(1-x_j)}}{e^{ta^j}-1} t^s \frac{dt}{t}.$$

$\underline{n=1}$: (Hurwitz zetas)

denom $= \underbrace{e^{ta^j}-1}_{\text{isolated } 0 \text{ at } t=0}$.

$n>1$ denom $= e^{t_1 a_1^j + t_2 a_2^j + \dots + t_n a_n^j} - 1$

– zero along a hyperplane through 0.

$C(\infty,\varepsilon)^n \supset$ polydisk of rad. ε

Integrand is not holo. on $C(\infty,\varepsilon)^n$ for any $\varepsilon>0$!!

<u>Shintani's fix.</u>

$$(0,\infty)^n = \dot{\bigcup_{i=1}^{n}} D_i$$

$$D_i = \{t : t_i \geq t_r \ \forall r=1,\dots,n\}.$$

on D_i: $t = u(y_1,\dots,y_{i-1},1,y_{i+1},\dots y_n)$.

$(y_r = \frac{t_r}{u}, r\neq i)$ $y_r \in (0,1) \ \forall r\neq i$.

$$\int_{(0,\infty)^n} \cdot = \int_{(0,\infty)} \int_{(0,1)^{n-1}}$$

$$J_\varepsilon = \int_{C(\infty,\varepsilon)} u^s \frac{du}{u} \int_{C(1,\varepsilon)} \prod_{j=1}^{d} \frac{e^{u y a^j (1-x_j)}}{e^{u y a^j} - 1} \prod_{r \neq i} y_r^s \frac{dy_r}{y_r}$$

ε — 1

$C(1,\varepsilon)$.

$$e^{u y a^j} - 1 = e^{u(y_1 a_1^j + \cdots + a_i^j + \cdots + y_n a_n^j)} - 1.$$

$a_i^j > 0$.

$u \in C(\infty,\varepsilon)$ $|u| \geq \varepsilon$. Consider $|y_r| = \varepsilon \; \forall r$.

This contour integral is holo on contour for ε small enough.

$$z_i(a,x,s) = \frac{1}{\Gamma(s)^n (e^{2\pi i n s} - 1)(e^{2\pi i s} - 1)^{n-1}} J_\varepsilon.$$

$$\boxed{\zeta(a,x,s) = \sum_{i=1}^{n} z_i(a,x,s)}$$