The Rank One Abelian Stark Conjecture

Lecture 1: Statement of the Conjecture

Stark's conjectures relate:

values of derivatives of partial zeta functions at $s=0$

to

(logarithms of absolute values of) units in algebraic number fields

— Analogous to (and a refinement of) the Dirichlet class # formula

— A version for units of the BSD Conjecture for elliptic curves (in fact, both are simultaneously generalized and refined in the ETNC)

- "Abelian" indicates we'll consider extensions K/F that are Galois with <u>abelian</u> galois group
- "Rank One" means we'll consider <u>first</u> derivatives of ζ-functions (conjectures very precise here)
- Rubin later made a similar precise conjecture in the higher rank case.

Ex $f \geq 2$, $f \in \mathbb{Z}$, $a \in \mathbb{Z}$, $(a, f) = 1$

Defn $\zeta_f(a, s) = \sum_{\substack{n=1 \\ n \equiv a \pmod f}}^{\infty} \frac{1}{n^s}$ $\quad s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$

Essentially a Hurwitz ζ-function.

$$\zeta_{Hur}(x, s) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s} \qquad x, s \in \mathbb{C},\ \mathrm{Re}(x) > 0,\ \mathrm{Re}(s) > 1.$$

$$\zeta_f(a, s) = f^{-s} \zeta_H\left(\frac{a}{f}, s\right)$$

$(0 < a < f)$

$$\zeta_f(a, s) = \frac{1}{f} \frac{1}{s-1} + b(a, f) + O(s-1) + \dots$$

$b(a,f) = ?$

Using functional equation, move from $s=1$ to $s=0$

Define $\zeta_f^+(a,s) = \zeta_f(a,s) + \zeta_f(-a,s)$

For $0<a<f$, we have

$$\zeta_f(a,0) = \frac{1}{2} - \frac{a}{f} \Rightarrow \zeta_f^+(a,0) = 0$$

$$\zeta_f^+(a,s) = c(a,f)\cdot s + \cdots \quad (\text{at } s=0)$$

$$\frac{d}{ds}\zeta_H(x,s)\Big|_{s=0} = \log\Gamma(x) - \frac{1}{2}\log 2\pi$$

$$\Rightarrow c(a,f) = \log\frac{\Gamma(\frac{a}{f})\Gamma(1-\frac{a}{f})}{2\pi}$$

$$= -\log\left(2\sin\left(\frac{\pi a}{f}\right)\right)$$

$$= -\frac{1}{2}\log\left(2-2\cos\left(\frac{2\pi a}{f}\right)\right)$$

$$= -\frac{1}{2}\log(u(a,f))$$

$$u(a,f) = (1-\zeta_f)(1-\zeta_f^{-1})$$

where $\zeta_f = e^{2\pi i a/f}$

$$u(a,f) \in K = \mathbb{Q}(\zeta_f)^+ = \mathbb{Q}(\zeta_f + \zeta_f^{-1})$$

Exer: $u(a,f) \in \mathcal{O}_K[\frac{1}{f}]^*$

in fact $u(a,f) \in \mathcal{O}_K^*$ if

f is div by ≥ 2 distinct primes.

Summary: $\zeta_f^+(a,0) = 0$

and $(\zeta_f^+)'(a,0) = -\frac{1}{2}\log u(a,f)$

where $u(a,f) \in \begin{cases} \mathcal{O}_K^* & f \text{ div by } \geq 2 \text{ primes} \\ \mathcal{O}_K[\frac{1}{f}]^* & f \text{ prime power} \end{cases}$

where $K = \mathbb{Q}(\zeta_f)^+$

The conjecture

K/F = abelian extension of # fields.

S = finite set of places of F

$\supset \{$archimedean places of F, $\mathfrak{p}$ ramifying in $K\}$

Assume $|S| \geq 3$. ($|S| = 2$ ok, see notes)

$$\zeta_{K/F,S}(\sigma, s) = \sum_{\substack{\mathfrak{n} \subset \mathcal{O}_F \\ (\mathfrak{n}, S) = 1,\ \sigma_{\mathfrak{n}} = \sigma}} \frac{1}{N\mathfrak{n}^s}, \quad \begin{array}{l} s \in \mathbb{C} \\ Re(s) > 1 \end{array}$$

$\sigma \in Gal(K/F)$

[When $F = \mathbb{Q}$, $K = \mathbb{Q}(\zeta_f)^+$, $\sigma = \sigma_a : \zeta_f \mapsto \zeta_f^a$
$S = \{\infty, p|f\}$, $\zeta^{*}_{K/F}(\sigma, s) = \zeta^{+}_f(a, s)$]

Let v be a place of F

Let $U_v = U_v(K) = \{ u \in K^* : |u|_w = 1 .$
$\forall \; w \nmid v \}$

"Strong v-units" (w can be archimedean.)

Fix $v \in S$.

Assume v splits completely in K

Let w be a place of K above v

v splits in $K \Rightarrow \zeta_{K/F,S}(\sigma, 0) = 0$

<u>Conj</u> $\exists \; \overset{= u(w).}{u} \in U_v(K)$ s.t. $|u|_{\sigma^{-1}(w)}$

$$\zeta'_{K/F,S}(\sigma, 0) = -\frac{1}{e} \log \overset{=}{|u^{\sigma}|_w}$$

$\forall \; \sigma \in Gal(K/F)$.

$e = \# \mu(K)$.

$u^{\sigma} = \sigma(u)$.

Furthermore, $K(u^{1/e})/F$ is abelian

Note: $|u|_{w'}$ is specified $\forall\, w'$ of K

So u unique up to mult by $\mu(K)$.

Smooth to get a unique

unit.

$T = \{\mathfrak{l}\}$ prime ideal $\mathfrak{l} \subset \mathcal{O}_F$

$\mathfrak{l} \notin S$

$\mathrm{Char}(\mathcal{O}_F/\mathfrak{l}) = \ell \geq [F:\mathbb{Q}] + 2$

$$\zeta_{K/F,S,T}(\sigma, s) = \zeta_{K/F,S}(\sigma, s) - N\mathfrak{l}^{1-s}\zeta_{K/F,S}(\sigma\sigma_{\mathfrak{l}}^{-1}, s).$$

(equivalent) Conj $\exists\, u_T \in U_{v,T}$

$$U_{v,T} = \{u \in U_v : u \equiv 1 \ (\mathrm{mod}\ \mathfrak{l}\mathcal{O}_K)\}$$

s.t. $\zeta'_{K/F,S,T}(\sigma, 0) = -\log|u_T^\sigma|_w$

u, u_T related: $\lambda = u^{1/e}$, $u_T = \dfrac{\lambda}{\sigma_{\mathfrak{l}}^{-1}(\lambda)^{N\mathfrak{l}}}$

u_T unique.

If $S \supset$ two places v, v' that split completely in K then $\zeta'_{K/F,S}(\sigma, 0) = 0$

$\Longrightarrow$ $u = 1$ works in conjecture

So we only consider:

- Case TR_∞: F = totally real. v = real. Places of K above v are real, all other archimedean places are complex.
- Case ATR: F = "almost totally real", i.e. F has <u>exactly</u> 1 complex place v. K = totally complex.
- Case TR_p: F = totally real, v = finite place. K = totally complex

Case TR_∞:

$$u^\sigma = e^{-2\zeta'_{K/F,S}(\sigma,0)}$$

$\downarrow$

Hilbert's 12^{th} problem

Case ATR:

$$|u_T^\sigma|_w = e^{-\zeta'_{K/F,S,T}(\sigma,0)}$$

$\in \mathbb{C}$

$\arg(u_T^\sigma) = ?$

<u>Motivating Question</u>: Can we give an exact formula for $u_T^\sigma \in K_w$, not just its absolute value?

Yes ATR: Ren-Sczech,
Charollois - Darmon

TR_p: Darmon - D.
Chapdelaine
D
Charollois - D.

Idea: ζ-functions are not the whole story on the analytic side. ζ-functions are merely shadows of "bigger" or "more refined" objects (cohomology classes / Shintani zeta-functions)

Main obstruction: units in ground field F.

Case TR_p.

$v = \mathfrak{p} \subset \mathcal{O}_F$, prime ideal.

$w = \mathfrak{P}$ above $\mathfrak{p}$

Let $R = S - \{\mathfrak{p}\}$

$$\zeta_{K/F,S,T}(\sigma,s) = (1 - N\mathfrak{p}^{-s}) \cdot \zeta_{K/F,R,T}(\sigma,s)$$

$$\zeta'_{K/F,S,T}(\sigma,0) = (\log N\mathfrak{p})\, \zeta_{K/F,R,T}(\sigma,0)$$

$$-\log|u_T^\sigma|_{\mathfrak{P}} = (\log N\mathfrak{p}) \cdot \underbrace{\mathrm{ord}_{\mathfrak{P}}(u_T^\sigma)}_{\in \mathbb{Z}}$$

Stark's conj: $\exists\ u_T \in \mathcal{U}_{\mathfrak{p},T}$

s.t. $\mathrm{ord}_{\mathfrak{P}}(u_T^\sigma) = \zeta_{K/F,R,T}(\sigma,0)$

Note: RHS $\in \mathbb{Z}$ by work of Deligne-Ribet / Pi. Cassou-Nogues / Barsky.