① Computing Chow-Heegner Points attached to diagonal cycles.

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(Darmon-Rotger 4)

$$V = X \times X \times X = X_1 \times X_2 \times X_3$$

$$X_1 \times X_2 \times X_3 \dashrightarrow X \qquad X_{12} \times X_{34} \subseteq X_1 \times X_2 \times X_3 \times X_4$$

$$\begin{tikzcd} \Phi \quad CH^2(X^3)_0 \arrow[r] \arrow[dr, bend right, "\Phi_E"'] & \mathrm{Jac}(X) \arrow[d, "\varphi_E"] \\ & E \end{tikzcd}$$

~~Given~~ $\cancel{T \subseteq CH^2(X_1 \times X_2 \times X_3)_0,}$ ~~compute~~

Given $\quad T \subseteq \mathrm{Pic}(X_1 \times X_2) \rightsquigarrow \Delta_T \in CH^2(X^3)_0$

$$\begin{tikzcd} & X_1 \times \cdots \times X_4 \arrow[dl, "\pi_{123}"'] \arrow[dr, "\pi_4"] & \\ X_1 \times X_2 \times X_3 & & X_4 \end{tikzcd}$$

<u>Goal</u> Compute $\Phi(\Delta_T)$?

<u>Key Remark</u>: $\Phi(\Delta_T) = \pi_4\left(\pi_{123}^{-1}(\Delta_T) \cdot (X_{12} \times X_{34})\right)$

is not useful for calculation.

② Analytic formula:

$$\begin{array}{ccc} CH^2(X^3)_0 & \xrightarrow{AJ} & \mathrm{Fil}^2 H^3_{dR}(X^3)^\vee / H_3(X^3(\mathbb{C}),\mathbb{Z}) \\ \downarrow \Phi & & \downarrow \\ \mathrm{Jac}(X) & \xrightarrow{AJ} & \Omega^1(X)^\vee / H_1(X(\mathbb{C}),\mathbb{Z}) \\ \downarrow p_E & & \downarrow ev_{\omega_E} \\ E & \longrightarrow & \mathbb{C}/\Lambda_E \end{array}$$

$$\boxed{\Phi_E(\Delta_T) = AJ(\Delta_T)(cl(\Delta_{12}) \otimes \omega_E)}$$

- $cl(\Delta_{12}) \in \mathrm{Fil}^1 H^2_{dR}(X_1 \times X_2) \cap H^2_B(X_1 \times X_2, \mathbb{Z})$
- $\omega_E \in \mathrm{Fil}^1 H^1_{dR}(X)$

③

A variant of this formula

$$\Delta_{GKS} = \Delta_{123} - \underset{\substack{\| \\ \{(x,x,p)\}}}{\Delta_{12}} - \Delta_{13} - \Delta_{23} + \underset{\substack{\uparrow \\ \{(x,p,p)\}}}{\Delta_1} + \Delta_2 + \Delta_3$$

$$\in CH^2(X^3)_0$$

$$\boxed{AJ(\Delta_T)(cl(\Delta_{12}) \otimes \omega_E) = AJ(\Delta_{GKS})(cl(T) \otimes \omega_E)}$$

Proof: Formal calculation.

$$cl(T) \in H^2_{dR}(X_1 \times X_2) = H^2_{dR}(X_1) \otimes H^0(X_2)$$
$$\oplus H^1_{dR}(X_1) \otimes H^1_{dR}(X_2)$$
$$\oplus H^0 \otimes H^2$$

(4)

Assume WLOG, $cl(T) \in H^1_{dR}(X_1) \otimes H^1_{dR}(X_2)$

Formula for $\underline{AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \omega_E)}$

<u>Iterated Integrals</u> (Chen; Hain, ...)

Given $\gamma : [0,1] \longrightarrow X(\mathbb{C})$

ω, η closed smooth 1-forms on $X(\mathbb{C})$

$$\int_\gamma \omega \cdot \eta := \int_{0 \le t_2 \le t_1 \le 1} \gamma^*\omega(t_1) \cdot \gamma^*\eta(t_2) = \int_\gamma \omega \cdot F_\eta$$

F_η = primitive of η on $\tilde{X}$ = universal covering space of $X(\mathbb{C})$

$$F_\eta(\gamma) = \int_\gamma \eta$$

⑤ simplifying assumption: $\langle \omega, \eta \rangle = 0$

$$\Rightarrow \boxed{(\omega \wedge \eta)\big|_{X_{12}} = d\alpha,} \text{ where } \alpha \text{ is of type } (1,0)$$

<u>Theorem</u>: $AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \omega_E) =$

$$\int_{\gamma_E} \omega \cdot \eta - \int_{\gamma_E} \alpha ,$$

$\gamma_E \in H_1(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C}$ is the Poincaré dual to ω_E

$$\int_{\gamma_E} \xi = \langle \omega_E, \xi \rangle \quad ☺$$

Proof: Later.

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<u>Problem.</u> α is not easy to calculate in practice.

<u>Exception</u> If $cl(T) = \omega \otimes \eta + \eta \otimes \omega$

$$cl(T)|_{X_{12}} = \omega \wedge \eta + \eta \wedge \omega = 0 \qquad \boxed{\alpha = 0.}$$

$$AJ(\Delta_{GKS})\big((\omega\otimes\eta + \eta\otimes\omega)\otimes\omega_E\big) = \int_{\gamma_E} \omega.\eta + \eta.\omega$$

$$= \left(\int_{\gamma_E} \omega\right)\left(\int_{\gamma_E} \eta\right)$$

Because we can't get our hands on α,
we will work with $H^1_{dR}(X) = \Omega^1_{\mathrm{II}}(X)/d\mathbb{E}(X)$, $\square$

<u>Notations</u> X = projective curve
$Y = X - \{\infty\}$

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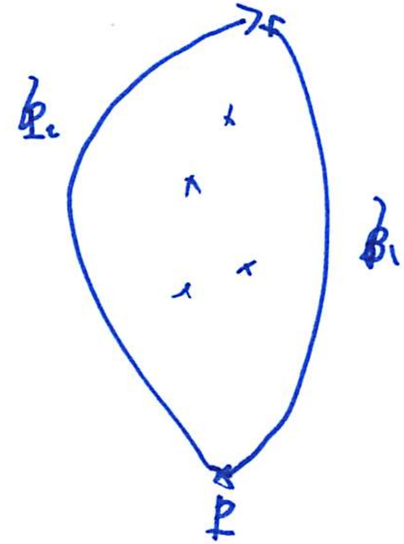
Assume, WLOG,

$\omega \in \Omega^1(X)$, $\eta \equiv$ meromorphic differential of second kind on X.

F_η = meromorphic primitive of η on $\hat{X}$.

$$F_\eta(z) = \int_{z_0}^{z} \eta.$$



ωF_η = meromorphic diff on $\hat{X}$.

Def Principal parts $\alpha \in \Omega^1_{mer}(\hat{X})$

$PP_x(\alpha) =$

$$\alpha_x = \sum_{i=-M}^{\infty} a_i t_x^i \cdot dt_x \qquad PP_x(\alpha) = \left(\sum_{j=-M}^{-1} a_j t_x^j\right) dt_x$$

Lemma For all $x \in \hat{X}$, $PP_x(\omega F_\eta)$ depends only on image of x in $X(\mathbb{C})$.

Proof Exercise.

(8) It makes sense to write $\mathcal{PP}_x(\omega F_\eta)$, $x \in X(\mathbb{C})$.

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<u>Proposition</u> There exist $\alpha \in \Omega^1_{mer}(X)$ s.t.

(1) $\mathcal{PP}_x(\alpha) = \mathcal{PP}_x(\omega F_\eta) \quad \forall\ x \in Y$

(2) $\mathcal{PP}_\infty(\alpha) = \mathcal{PP}_\infty(\omega F_\eta) \quad \left(\text{mod } \frac{dq}{q}\right)$

$$\boxed{\alpha \text{ well-defined mod } \Omega^1(X).}$$

<u>Theorem</u> $$AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \omega_E) = \int_{\gamma_E} \omega F_\eta - \int_{\gamma_E} \alpha$$

$$= \int_{\gamma_E} \omega \bullet \eta - \int_{\gamma_E} \alpha$$

Algorithm for Chow-Heegner Points

Set-Up
- $X = X_0(N)$
- $Y = X_0(N) - \{\infty\}$
- $\varphi_E : X_0(N) \longrightarrow E. \quad \longleftrightarrow \quad f \in S_2(\Gamma_0(N))$

- $cl(T) = \sum \omega_i \otimes \eta_i$

Example g = eigenform of ~~X_0~~ wt 2 on $\Gamma_0(N)$ with rational coefficients. $g \neq f$.

- $\omega_g \in \Omega^1(X)^g$
- $\eta_g \in H^1_{dR}(X)^g \setminus \mathbb{C}\omega_g$.

FACT ~~cl~~ $\omega_g \otimes \eta_g - \eta_g \otimes \omega_g$ is a Hodge class in $H^2_{dR}(X_1 \times X_2)$

i.e., it belongs to $\mathrm{Fil}^1 H^2_{dR}(X_1 \times X_2) \cap H^2_B(X_1 \times X_2, \mathbb{Z})$.

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Step 1 $P(f,g) = AJ(\Delta_{GKS})\left((\omega_g \otimes \eta_g - \eta_g \otimes \omega_g) \otimes \omega_f\right)$

$\in \mathbb{C}/\Lambda_E = E(\mathbb{C})$

Step 1 compute q-exp for ω_f, ω_g.

$f = \sum_{n=1}^{\infty} a_n(f) q^n$

$g = \sum \cdots$

Step 2 compute q-exp of η_g

One approach $\omega_1, \ldots, \omega_t$ = basis of $\Omega^1(X) = S_2(\Gamma_0(N))$

- Let $u \in \mathcal{O}_Y$ with $\mathrm{ord}_\infty(u)$ as small as possible.

$u\omega_1, \ldots, u\omega_t$

"Claim": $(\omega_1, \ldots, \omega_t, u\omega_1, \ldots, u\omega_t)$ is a basis for $H^1_{dR}(X)$.

$\rightsquigarrow \eta_g$ using modular symbols.

Step 3 Find a meromorphic diff α on X, regular on Y,

st $PP_\infty(\alpha) = PP_\infty(2\omega_g F\eta_g) \pmod{\frac{dq}{q}}$

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Step 4 Compute $\gamma_E \in H_1(X_0(N), \mathbb{C})^f$

$\rightarrow$ Modular symbol algorithm

Step 5 $\quad P(f,g) = 2\int_{\gamma_E} \omega_g \cdot \eta_g - \int_{\gamma_E} \alpha$