V. Rotger March 13, 2011

(Darmon-Rotger Lec. 3)

Explicit computation of Chow-Heegner points associated to

$$V = X_0(N) \times X_0(N) \times X_0(N)$$

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or

Making explicit

$$\begin{tikzcd}
CH^2(V)_0 \arrow[r, "AJ"] \arrow[d, "\phi"] & J^2(V) \arrow[d, "\phi^{an}"] \\
\mathrm{Jac}\, X_0(N) \arrow[r, "\cong"] \arrow[d, "\pi_f"] & \dfrac{\Omega^1(X_0(N))^\vee}{H_1(X_0(N),\mathbb{Z})} \arrow[d, "\pi_f^{an}"] \\
E_f \arrow[r, "\sim"] & \mathbb{C}/\Lambda_f
\end{tikzcd}$$

$$f = \sum a_n(f)\, q^n \in S_2(\Gamma_0(N)) \text{ newform}$$

$$[\mathbb{Q}_f = \mathbb{Q}(\{a_n(f)\}) : \mathbb{Q}] = d_f < +\infty$$

Geometry of f: $\quad \omega_f = 2\pi i f(z)\, dz \in \Omega^1(X)$

where $X = X_0(N)$

$$\begin{array}{ccccc} \mathrm{Jac}(X) = \mathrm{Pic}(X)_0 = CH^1(X)_0 & \circlearrowleft & \mathbb{T} = \langle T_p, p \nmid N \rangle_{\mathbb{Q}} & & \\ \downarrow \text{def} & & \downarrow & & T_p \\ & & & & \downarrow \\ A_f & \circlearrowleft & \mathbb{Q}_f & & a_p(f) \end{array}$$

be the maximal quotient on which $\mathbb{T}$ acts through $\mathbb{Q}_f$. Fact: $\dim(A_f) = d_f$.

L-function of f: $\quad L(f,s) = \prod_p \Big(\underbrace{(1-\alpha_p(f)\, p^{-s})(1-\beta_p(f)\, p^{-s})}_{\text{Poly's of deg 2 in } p^{-s}} \Big)^{-1}$

Thm: $L(f,s)$ can be "completed" to

$$\Lambda(f,s) = L(f,s) \cdot L_\infty(s) \quad \text{so that:}$$

1. $\Lambda(f,s)$ admits analytic cont'n to $\mathbb{C}$
2. $\Lambda(f,s) = \varepsilon \cdot \Lambda(2-s)$, $\varepsilon = \pm 1$

BSD: $\quad \mathrm{ord}_{s=1} L(f,s) = \mathrm{rank}_{\mathbb{Q}_f}(\mathbb{Q} \otimes A_f(\mathbb{Q}))$

$f_1, f_2, f_3 \in S_2(\Gamma_0(N))$ newforms

$$\mathbb{Q}_{f_1,f_2,f_3} = \mathbb{Q}_{f_1} \cdot \mathbb{Q}_{f_2} \cdot \mathbb{Q}_{f_3}$$

$$L(f_1 \otimes f_2 \otimes f_3, s) = \prod_{p \nmid N} \Big(\underbrace{\big(1 - \alpha_p(f_1)\alpha_p(f_2)\alpha_p(f_3)p^{-s}\big) \cdot \ldots}_{\text{deg 8 poly in } p^{-s}} \Big)^{-1}$$

Thm: Can be completed to $\Lambda(f_1 \otimes f_2 \otimes f_3, s)$:

1. Λ admits meromorphic cont'n to $\mathbb{C}$
2. $\Lambda(s) = \varepsilon \Lambda(4-s)$ $\quad c = 2$, $\varepsilon = -\prod_{p \mid N} \varepsilon_p$

BSD: $\operatorname{ord}_{s=2} \Lambda(s) = \operatorname{rank}_{\mathbb{Q}(f_1,f_2,f_3)} \mathbb{Q} \otimes CH^2(X^3)_0 [f_1, f_2, f_3]$

Assume "Heegner hyp" / "Gross-Prasad hyp" : $\varepsilon_p = +1 \Rightarrow \varepsilon = -1$

$\Rightarrow \operatorname{ord}_{s=2} \Lambda(s)$ is odd ≥ 1

$\Rightarrow$ Expect to find $0 \neq \Delta \in CH^2(X^3)_0 [f_1, f_2, f_3]$

See work of Yuan - Zhang - Zhang.

Let $\Pi = X_{12} \times X_{34} \quad \in CH^2(X_1 \times X_2 \times X_3 \times X_4)$

$$\begin{tikzcd}
\phi: & CH^2(X^3)_0 \arrow[r] & CH^1(X)_0 \\
& \Delta \arrow[r, mapsto] \arrow[d, bend right, "AJ"'] & \pi_{123}^*(\Delta)\cdot\Pi \arrow[r, mapsto, "\pi_{4,*}"] & P_\Delta \arrow[d, bend left, "AJ"] \\
\phi^{an}: & J^2(X^3) = \dfrac{Fil^2 H^3(X^3)^\vee}{\text{Lattice}} \arrow[rr] & & \dfrac{\Omega^1(X)^\vee}{H_1(X,\mathbb{Z})}
\end{tikzcd}$$

$\Delta \in CH^2(X^3)_0$, $P_\Delta \in CH^1(X)_0$

$\underbrace{\pi_{123}^*(\Delta)}_{\text{codim 2 in } X^4} \cdot \underbrace{\Pi}_{\text{codim 2}}$: bunch of points

$$\begin{tikzcd}
& X^4 \arrow[dl] \arrow[dr] & \\
X_1 \times X_2 \times X_3 & & X_4
\end{tikzcd}$$

$$Fil^2 H^3(X^3) \longleftarrow \Omega^1(X)$$
$$cl(X_{12}) \otimes \rho(z_3) \longleftarrow\!\!\shortmid\; \rho(z_4)$$

Recall

$$CH^1(X_1 \times X_2) \xrightarrow{cl} H^2(X_1 \times X_2)$$
$$X_{12} \longmapsto [X_{12}] \text{ "=" } cl(X_{12})$$

Now

$$\pi_f(P_\Delta) = AJ(\Delta)(cl(X_{12}) \otimes \rho_f) \in \mathbb{C}/\Delta_f$$

$\pi_f(P_\Delta) \in E_f$

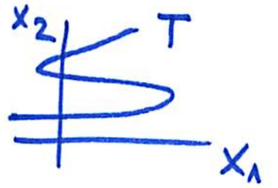
$$AJ(\Delta)(cl(X_{12}) \otimes \rho_f) := \int_{\partial^{-1}\Delta} cl(X_{12}) \otimes \rho_f$$

Construct cycles $\Delta \in CH^2(X^3)_0$

We construct a map $\mathrm{Pic}(X_1 \times X_2) \to CH^2(X^3)_0$

$$\underset{X_{12},\ T_p}{\mathrm{Pic}(X_1 \times X_2)} \qquad \psi = \psi_2 \circ \psi_1$$

$T \subset X_1 \times X_2$



$$\varphi_T : T \xrightarrow{pr_1} X_1 \overset{Id}{=} X_3$$

$$\Delta_T = \underset{\substack{\cap\! | \\ T \times X_3}}{\mathrm{graph}(\varphi_T)} - T \times 0_3 - d\,(0_1 \times 0_2 \times X_3) \in CH^2(X_1 \times X_2 \times X_3)$$

$$d = \deg(\varphi_T)$$

$$\varepsilon : CH^2(X_1 \times X_2 \times X_3) \longrightarrow CH^2(X_1 \times X_2 \times X_3)$$

$$\Delta \longmapsto \Delta - \pi_{13,*}(\Delta) - \pi_{23,*}(\Delta)$$

$$\psi(T) = \varepsilon(\Delta_T)$$

<u>Example:</u> $T = X_{12} \longmapsto X_{123} - X_{12} - X_3 = \Delta_T$

$$\begin{aligned}\psi(T) = \varepsilon(X_{123} - X_{12} - X_3) = {} & X_{123} - X_{13} - X_{23} \\ & - (X_{12} - X_1 - X_2) \\ & - (X_3 - X_3 - X_3)\end{aligned}$$

$$\Delta_{GKS} := X_{123} - X_{12} - X_{13} - X_{23} + X_1 + X_2 + X_3$$

Thm (A) For all $T \subseteq X_1 \times X_2$, $\varepsilon\Delta_T \in CH^2(X^3)_0^{\otimes}$ (6)

(B) There is a formula for computing
$AJ(\varepsilon\Delta_T)$ in terms of path (iterated) integrals.

(B1) $P_T := AJ(\varepsilon\Delta_T)(cl(X_{12}) \wedge \rho_f) = \underline{\underline{AJ(\Delta_{GKS})}}(cl\, T \wedge \rho_f)$

(B2) Replace T by $T - \pi_{1,*}T - \pi_{2,*}T$, which is harmless.

so that $cl(T) \in H^1(X_1) \otimes H^1(X_2) \in H^2(X_1 \times X_2)$

$$\cup\!\!|$$

$$\in \Omega^1(X_1) \otimes H^1(X_2) + H^1(X_1) \otimes \Omega^1(X_2)$$

$cl(T) = \sum \omega_i(z_1) \otimes \eta_i(z_2) + \sum \eta_j(z_1) \otimes \omega_j(z_2)$

where $\omega_i, \omega_j \in \Omega^1(X)$, $\eta_j, \eta_i \in H^1(X)$

$$= \sum \underline{\underline{\omega_i(z_1) \otimes}} \eta_i(z_2) + \sum \omega_j(\underline{z_1) \otimes} \eta_j(z_2) -$$

$$- \sum \left(\omega_j(z_1) \otimes \eta_j(z_2) + \eta_j(z_1) \otimes \omega_j(z_2)\right)$$

We should be able to compute

$AJ(\Delta_{GKS})(\omega \otimes \eta \otimes \rho) = ?_2$

and $AJ(\Delta_{GKS})((\omega \otimes \eta + \eta \otimes \omega) \otimes \rho) = ?_1$

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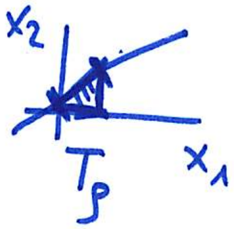
$$?_1 = \int_{\gamma_\rho} \omega \cdot \int_{\gamma_\eta} \eta$$

where γ_ρ is the Poincaré dual of ρ :
For any $\gamma \in H_1(X,\mathbb{Z})$, $\gamma \cdot \gamma_\rho = \int_\gamma \rho$

$$\int_{T_\rho} \omega(z_1) \wedge \eta(z_2) \underset{\text{Chen's notation}}{=:} \int_{\gamma_\rho} \omega(z) \bullet \eta(z)$$



$\gamma : [0,1] \longrightarrow X(\mathbb{C})$

$$= \iint_{0 \le s \le t \le 1} \gamma^* \omega(s)\, \gamma^* \eta(t)$$

$$= \int_0^1 \left(\int_0^t \gamma^* \omega(s) \right) \cdot \gamma^* \eta(t) .$$

$$?_2 = \int_{\gamma_\rho} \omega(z) \bullet \eta(z) - \int_{\gamma_\rho} \alpha(z)$$

where $\qquad d\alpha(z) = \omega(z) \wedge \eta(z)$.

Need that $\qquad H^1(X) \otimes H^1(X) \longrightarrow H^2(X)$

$$\omega \otimes \eta \longmapsto \omega \wedge \eta = 0$$