OUTLINE AND REFERENCES FOR PROJECT: HASSE PRINCIPLE FOR RATIONAL FUNCTION FIELDS, AWS 2009

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1. Introduction

Hasse-Minkowski's theorem asserts that a quadratic form over a number field k admits a nontrivial zero if it does over completions at all places of k. One could look for analogues of Hasse principle for function fields. Let k be a field of characteristic not 2 and Ω a set of discrete valuations of k. Let $\widehat{k_v}$ denote the completion of k at v. We say that k satisfies Hasse principle with respect to Ω if every quadratic form over k which is isotropic over $\widehat{k_v}$ for all $v \in \Omega$ is isotropic. We say that k satisfies weak Hasse principle with respect to Ω if every quadratic form over k which is hyperbolic over $\widehat{k_v}$ for all $v \in \Omega$ is hyperbolic.

Let $k(t)$ be the rational function field in one variable over k. Let V denote the set of all discrete valuations of $k(t)$ trivial on k. We shall discuss analogues of Hasse principle for isotropy of quadratic forms over $k(t)$ with respect to V. An affirmative answer to the Hasse principle for $k = Q_p$ would lead to the fact that every quadratic form in at least 9 variables over $Q_p(t)$ has a nontrivial zero.

2. Complete discrete valuated fields

For a quick introduction to quadratic forms over rational function fields, we refer to [EKM], chapter III.

Let K be a field with a complete discrete valuation v. Let $\mathcal{O}_v$ be the ring of integers in K, π_v a uniformizing parameter for v and $\kappa_v = \mathcal{O}_v/\langle \pi_v \rangle$ the residue field at v. We assume that the characteristic of κ_v is not 2.

By Hensel's lemma, the square classes of K^* are given by $\{u_\alpha, \pi_v u_\alpha,\ \alpha \in I\}$, where $\{\bar{u}_\alpha,\ \alpha \in I\}$ is the set of square classes in κ_v^*; bar denotes reduction modulo π_v and $u_\alpha \in \mathcal{O}_v$ are a set of representatives for $\bar{u}_\alpha$, $\alpha \in I$. Thus every quadratic form over K is isometric to a diagonal

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form $\langle u_1, \cdots, u_r\rangle \perp \pi_v\langle v_1, \cdots, v_s\rangle$, u_i, v_j, units in $\mathcal{O}_v$. The quadratic forms $q_1 = \langle \bar{u}_1, \cdots, \bar{u}_r\rangle$ and $q_2 = \langle \bar{v}_1, \cdots, \bar{v}_s\rangle$ over κ_v are determined in $W(\kappa_v)$ uniquely by q and do not depend on specific diagonalisation chosen. One has two residue homomorphisms

$$\delta_v^1 : W(K) \to W(\kappa_v), \quad \delta_v^2 : W(K) \to W(\kappa_v)$$

given by $\delta_v^1([q]) = [q_1]$, $\delta_v^2([q]) = [q_2]$. These are called the *first and second residue homomorphisms.*

Theorem 2.1. *(Springer) Let K be a complete discrete valuated field. Let $q = q_1 \perp \pi_v q_2$ be a quadratic form over K with $q_1 = \langle u_1, \cdots, u_r\rangle$, $q_2 = \langle v_1, \cdots, v_s\rangle$, u_i, v_j, units in $\mathcal{O}_v$. Then q is isotropic if and only if q_1 or q_2 is isotropic which in turn is equivalent to $\bar{q}_1$ or $\bar{q}_2$ is isotropic over κ_v. The residue homomorphisms yield an isomorphism*

$$W(K) \overset{(\delta_v^1, \delta_v^2)}{\to} W(\kappa_v) \oplus W(\kappa_v).$$

Corollary 2.2. *Let k be a field of chracteristic not 2 and $K = k((t))$. The map $W(k) \to W(k((t)))$ is a split injection with δ_v^1 giving a section.*

3. Milnor exact sequence

Let k be a field of characteristic not 2 and $K = k(t)$ the rational function field in one variable over k. Let V denote the set of all discrete valuations of $k(t)$ trivial on k. For $v \in V$, if $k[t] \subset \mathcal{O}_v$, v corresponds uniquely to a monic irreducible polynomial $\pi \in k[t]$ and $\mathcal{O}_v = k[t]_{(\pi)}$. If $k[t] \not\subset \mathcal{O}_v$, $\mathcal{O}_v = k[t^{-1}]_{(t^{-1})}$. Let $V_0 = \{v \in V,\ k[t] \subset \mathcal{O}_v\}$; $V = V_0 \cup \{v_{t^{-1}}\}$. For each $v \in V$, we have the residue homomorphisms $\delta_v^i : W(k(t)) \to W(\kappa_v)$ defined as the composite

$$W(k(t)) \to W(\widehat{k(t)}_v) \overset{\delta_v^1}{\to} W(\kappa_v),$$

$\widehat{k(t)}_v$ denoting the completion of $k(t)$ at v.

We note that if $v = v_\pi$, π a monic irreducible polynomial in $k[t]$, the residue field $\kappa_v = k[t]/\langle \pi\rangle$ is a finite extension of k and $\widehat{k(t)}_v \overset{\sim}{\to} \kappa_v((X))$, the field of Laurent series over κ_v.

Theorem 3.1. *(Milnor) We have an exact sequence of Witt groups*

$$0 \to W(k) \to W(k(t)) \overset{(\delta_v^2)}{\to} \bigoplus_{v \in V_0} W(\kappa_v) \to 0.$$

Corollary 3.2. *Let q be a quadratic form over $k(t)$ such that q is hyperbolic over $\widehat{k(t)}_v$ for each $v \in V_0$. Then q is hyperbolic.*

Proof. The class of q in $W(k(t))$ is in the kernel of (δ_v^2) and hence there is a quadratic form q_0 over k such that $[q_0] = [q]$ in $W(k(t))$. Over the completion at the t-adic valuation v_t of $k(t)$, $[q_0] = [q] = 0$. By (2.2), $[q_0] = 0$ in $W(k)$ and $[q] = 0$ in $W(k(t))$. □

Corollary 3.3 ([CTCS], Proposition 1.1)**.** *Let k be a number field and $\Omega(k)$ the set of all places of k. The map*

$$\eta : W(k(t)) \longrightarrow \prod_{w \in \Omega(k)} W(k_w(t))$$

has trivial kernel.

Proof. For $w \in \Omega(k)$ and $\tilde{v} \in V(k_w(t))$, if v denotes the restriction of $\tilde{v}$ to $k(t)$, then $v \in V(k(t))$ and $\kappa(\tilde{v})$ which is the residue field at $\tilde{v}$, is the completion of the residue field $\kappa(v)$ at a place extending w. In fact, every completion of $\kappa(v)$ at a place of $\kappa(v)$ is accounted for in this way. Further, the diagram

$$\begin{array}{ccc} W(k(t)) & \longrightarrow & W(k_w(t)) \\ \Big\downarrow{\scriptstyle \partial_v^2} & & \Big\downarrow{\scriptstyle \partial_{\tilde{v}}^2} \\ W(\kappa(v)) & \longrightarrow & W(\kappa(\tilde{v})) \end{array}$$

commutes. Thus if q is a quadratic form over $k(t)$ such that $\eta(q) = 0$, then $\partial_v^2(q) = 0$ for all $v \in V_0$. Thus q is the image of some q_0 in $W(k)$. In $W(k_w(t))$, $[q] = [q_0] = 0$; injectivity of $W(k_w) \to W(k_w(t))$ implies that $[q_0] = 0$ in $W(k_w)$. This is true for all $w \in \Omega(k)$. By Hasse-Minkowski's Theorem, $[q_0] = 0$ in $W(k)$; in particular $q = 0$ in $W(k(t))$. □

The above corollary is an analogue of weak Hasse principle for global fields: "locally" hyperbolic forms are hyperbolic.

4. Forms of dimension at most 4

Let k be a field of characteristic not 2 and $K = k(t)$. Let V be the set of discrete valuations of K trivial on k. A form q of dimension 2 is isotropic if and only if it is hyperbolic. Thus by weak Hasse principle a rank 2 form over K is isotropic if and only if it is isotropic over $\widehat{K}_v$ for all $v \in V$.

Since the property "isotropic" is insensitive to scaling, we may assume q represents 1. Let $q = \langle 1, a, b \rangle$ be the rank 3 quadratic form over K. Then q is isotropic if and only if $\tilde{q} = \langle 1, a, b, ab \rangle$ is isotropic. Further, every quadratic form q of dimension 4 and discriminant one is isometric to $\lambda\tilde{q}$ for some $\lambda, a, b \in K^*$. The form $\tilde{q}$ is the norm form from the quaternion algebra $H(-a,-b)$ and $\tilde{q}$ is isotropic if and only if $H(-a,-b)$ is split, i.e., it is isomorphic to $M_2(K)$. In this case, $\tilde{q}$ is hyperbolic. Thus weak Hasse principle for K yields: a form of dimension 3 or a form of dimension 4 and discriminant 1 is isotropic if and only if it is isotropic over $\widehat{K}_v$ for all $v \in V$.

If dimension of q is 4, there are counter examples to Hasse principle over $\mathbb{Q}(t)$, $\mathbb{Q}$ denoting the field of rational numbers.

Project A: *We shall work out some explicit counter examples to Hasse principle in dimension* 4 *over* $\mathbb{Q}$.

5. Rational function fields over p-adic fields

5.1. **Motivation.** Let V be the set of discrete valuations of $\mathbb{Q}_p(t)$ trivial on $\mathbb{Q}_p$. We call a set $\tilde{V}$ of discrete valuations of $\mathbb{Q}_p(t)$ geometric if there is a regular proper scheme $\tilde{X} \to Spec(\mathbb{Z}_p)$ such that $\tilde{V}$ = set of discrete valuations of $\mathbb{Q}_p(t)$ centered on codimension one points of $\tilde{X}$.

If $\mathcal{X}$ is a regular scheme and $x \in \mathcal{X}$ a codimension one point, the local ring $\mathcal{O}_{\mathcal{X},x}$ is a discrete valuation ring and we denote by v_x the discrete valuation in the function field of $\mathcal{X}$ with ring of integers $\mathcal{O}_{\mathcal{X},x}$.

Example 5.1. If $\tilde{X} = \mathbb{P}^1_{\mathbb{Z}_p} \xrightarrow{\eta} Spec(\mathbb{Z}_p)$, $\tilde{V} = V \cup \{v_p\}$ where v_p is the discrete valuation of $\mathbb{Q}_p(t)$ arising from the special fiber of η.

Given a geometric set $\tilde{V}$ of discrete valuations of $\mathbb{Q}_p(t)$, for each $v \in \tilde{V}$, the residue field κ_v of v is a p-adic field or a global field of positive characteristic and such a field has u-invariant 4 (see lecture notes). By Springer's theorem, $u(\widehat{\mathbb{Q}_p(t)}_v) = 8$. If there is Hasse principle for isotropy of quadratic forms over $\mathbb{Q}_p(t)$ with respect to some geometric set $\tilde{V}$ of discrete valuations of $\mathbb{Q}_p(t)$, then it would follow that $u(\mathbb{Q}_p(t)) = 8$. A new proof of $u(\mathbb{Q}_p(t)) = 8$ for $p \neq 2$ via Hasse principle is given in [CTPS]; this uses certain patching results for fields and linear algebraic groups due to [HH] and [HHK].

6. Dimension 4 case

Lemma 6.1. *Let F be any field and $q = \langle 1, a, b, abd\rangle$ be a* 4 *dimensional quadratic form over F with non trivial discriminant d. Then q is isotropic if and only if it is isotropic in the discriminant extension $F(\sqrt{d})$.*

Proof. Let $v, w \in F^4$ be such that $q(v + \sqrt{d}w) = 0$, v, w not both zero. If $w = 0$ then q is isotropic. Suppose $w \neq 0$ and $q(v) \neq 0$. Then we have $q(v) + dq(w) = 0$ and $b_q(v, w) = 0$. The subspace $Fv \oplus Fw$ in F^4 has dimension 2 and q restricted to this subspace is represented with respect to the basis $\{v, w\}$ by $\langle -d\alpha, \alpha\rangle$ where $q(v) = \alpha$. Thus $q \cong \alpha\langle 1, -d\rangle \perp q_1$ with $disc(q_1) = disc(q)(-d) = -1$. Hence q_1 is isotropic; thus q is also isotropic. $\square$

Let $q = \langle 1, a, b, abd\rangle$ be a dimension 4 quadratic form over $\mathbb{Q}_p(t)$ with discriminant $d \in \mathbb{Q}_p(t)^*/\mathbb{Q}_p(t)^{*2}$. If $d = 1$, we have already seen that Hasse principle holds with respect to V. Suppose $d \neq 1$. Let $L = \mathbb{Q}_p(t)(\sqrt{d})$.

The field $L = l(X)$ is the function field of a smooth projective curve X over l which is an extension of $\mathbb{Q}_p$ of degree at most 2. We denote by $Br(X)$ the subgroup of $Br(k(X))$ consisting of all classes $[A]$ *unramified* at every closed point $x \in X$, i.e., there is an Azumaya algebra A_x over $\mathcal{O}_{X,x}$ such that $[A_x \otimes_{\mathcal{O}_{X,x}} k(X)] = [A]$. [An algebra B over a local ring $(R, \mathfrak{m})$ is called *Azumaya* if $B \otimes_R R/\mathfrak{m}$ is a central simple algebra over $R/\mathfrak{m}$]. The form $q_L \cong \langle 1, a, b, ab\rangle$ is the quaternion norm form $H(-a, -b)$. The norm is hyperbolic over all completions $\widehat{L}_w$ at discrete valuations w of L given by closed points of the curve X since w extends some $v \in V$. Thus the algebra $H(-a, -b)$ is "unramified" at all codimension one points of X and hence $H(-a, -b) \in Br(X)$. There is a non-degenerate pairing [Li]

$$Br(X) \times PicX \xrightarrow{\eta} \mathbb{Q}/\mathbb{Z}$$

given by $\eta(\xi, x) = cor_{\kappa_x|l}(\xi_x)$, for a closed point $x \in X$. Here ξ_x is the specialisation of ξ at the closed point x which gives a class in $Br(\kappa_x)$ and $cor : Br(\kappa_x) \to Br(l)$ is the corestriction map. We identify $Br(l) \cong \mathbb{Q}/\mathbb{Z}$ via local class field theory. This non degenerate pairing yields the fact: If $\xi \in Br(X)$ and $\xi_{\widehat{L}_w} = 0$ for all completions $\widehat{L}_w$ at discrete valuations of L centered on closed points of X then $\xi = 0$. Thus $H(-a, -b) = 0$ in $Br(L)$ which implies that q_L is hyperbolic. By 6.1, q is isotropic.

7. Dimension 6 forms and a counter example to Hasse principle

Let F be a field of characteristic not 2. Let $A = H_1 \otimes H_2$ be a tensor product of two quaternion algebras $H_1 = H(a, b)$, $H_2 = H(c, d)$.

Such an algebra is called a biquaternion algebra. To A is associated a dimension 6 quadratic form of discriminant one

$$q = \langle -a, -b, ab, c, d, -cd \rangle$$

called the *Albert form*. In fact, every dimension 6 quadratic form over F of discriminant one is similar to an Albert form. We have the following:

Theorem 7.1. *Given a biquaternion algebra A, the similarity class of the Albert form q_A associated to A is uniquely determined. Further,*

1. *A is division if and only if q_A is anisotropic.*
2. *$A \cong M_2(H)$, H quaternion division over F if and only if q_A is isotropic but not hyperbolic.*
3. *$A \cong M_4(F)$ if and only if q_A is hyperbolic.*

There are examples in literature [Sa], [RTS] of biquaternion division algebras over $\mathbb{Q}_p(t)$.

Project B: *We shall analyse whether the Albert forms arising from biquaternion division algebras lead to a counter example to Hasse principle in dimension 6 over $\mathbb{Q}_p(t)$.*

8. Amer-Brumer theorem and counter examples in dimensions 6, 7 or 8

Let F be a field of characteristic not 2. We begin with the following theorem (cf. [EKM], pp 74).

Theorem 8.1. *(Amer-Brumer) Let f and g be two quadratic forms on a vector space V over F. The form $f + tg$ over $V \otimes F(t)$ is isotropic if and only if f and g have a common non trivial zero in V.*

Suppose f and g are quadratic forms over $\mathbb{Q}_p$ of dimension $n + 1$ with the following properties:

1. $n \geq 5$.
2. The variety $X_{f,g}$ which is the intersection of the two quadrics $f = 0$ and $g = 0$ in $\mathbb{P}^n$ has no $\mathbb{Q}_p$ rational point.
3. For each $\lambda \in \overline{\mathbb{Q}_p}$, the form $f + \lambda g$ has rank $\geq n$ and the form g has rank $\geq n$, $\overline{\mathbb{Q}_p}$ denoting an algebraic closure of $\mathbb{Q}_p$.
4. $\det(f + tg)$ is a separable polynomial, (i.e., has no multiple zeroes).

Then by Amer-Brumer theorem, the form $f+tg$ over $\mathbb{Q}_p(t)$ is anisotropic; further, at each completion $\widehat{\mathbb{Q}_p(t)}_v$, $v \in V$, $f + tg \cong q_1 \perp \pi_v q_2$ with $\dim(q_1) \geq n$. In fact, if λ is a zero of π_v in $\overline{\mathbb{Q}_p}$, over a completion of $\overline{\mathbb{Q}_p}(t)$ at $t-\alpha$, $\pi_v = (t-\lambda)h(t)$, $h(\lambda) \neq 0$, $q = q_1 \perp (t-\lambda)h(\lambda)q_2$ over the completion of $\overline{\mathbb{Q}_p}(t)$ at $v_{(t-\lambda)}$. Thus $f+\lambda g = q_1$ has rank $\geq n$ and q_1 is isotropic in the residue field at π_v. Hence q is isotropic over $\widehat{\mathbb{Q}_p(t)}_v$. Similar arguments work for the completion at $v_{t^{-1}}$. Thus $f+tg$ is isotropic over $\widehat{\mathbb{Q}_p(t)}_v$ for all $v \in V$ but is anisotropic.

It is easy to produce pairs of forms (f,g) of dimensions 6, 7 or 8 with property 2 satisfied: Let $\langle 1, -u, -p, up\rangle$ be the unique anisotropic quadratic form in dimension 4 over $\mathbb{Q}_p$. Let

$$\begin{aligned} n(X_1,X_2,X_3,X_4) &= X_1^2 - uX_2^2 - pX_3^2 + upX_4^2, \\ n_0(X_1,X_2,X_3) &= X_1^2 - uX_2^2 - pX_3^2. \end{aligned}$$

Both n and n_0 are anisotropic forms over $\mathbb{Q}_p$. Set

$$\begin{aligned} f_1 &= n(X_1,X_2,X_3,X_4), & g_1 &= n(Y_1,Y_2,Y_3,Y_4) \\ f_2 &= n(X_1,X_2,X_3,X_4), & g_2 &= n_0(Y_1,Y_2,Y_3) \\ f_3 &= n_0(X_1,X_2,X_3), & g_3 &= n_0(Y_1,Y_2,Y_3). \end{aligned}$$

Let $h_1 = f_1 + tg_1$, $h_2 = f_2 + tg_2$, $h_3 = f_3 + tg_3$. Since f_i, g_i are anisotropic over $\mathbb{Q}_p$, by Amer-Brumer theorem, h_i are anisotropic over $\mathbb{Q}_p(t)$. We have $\dim(h_1) = 8$, $\dim(h_2) = 7$, $\dim(h_3) = 6$. But h_i, $1 \leq i \leq 3$ do not have property 3.

$h_1(0) = f_1$ has rank 4, $h_2(0) = f_2$ has rank 4 and $h_3(0) = f_3$ has rank 3.

We construct pairs $(\tilde{f}_i, \tilde{g}_i)$, $1 \leq i \leq 3$ starting from the pairs (f_i, g_i), satisfying $1), 2), 3)$ and 4), by using a compactness argument which we outline below.

Let $\mathbb{P}^N$ denote the projective space of quadrics in $n+1$ variables. We have a closed set $Z \subset \mathbb{P}^n \times \mathbb{P}^N \times \mathbb{P}^N$ consisting of triples $\{(x,f,g) : f(x) = 0,\ g(x) = 0\}$. The variety Z admits a fibration $Z \to \mathbb{P}^n$ whose fibers are of the type $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Thus Z is smooth and geometrically integral.

Given two quadrics f, $g \in \mathbb{P}^N(\mathbb{Q}_p)$, the fiber of $Z \to \mathbb{P}^N \times \mathbb{P}^N$ at (f,g) is the variety

$$X(f,g) = \{x \in X,\ f(x) = 0,\ g(x) = 0\},$$

of intersection of the quadrics $f = 0 = g$. The dimensions of the fibers vary from point to point. There is a non empty Zariski open set U of

$\mathbb{P}^N \times \mathbb{P}^N$ such that the fiber at $(f,g) \in U$ has dimension $n-2$; i.e., the fiber $X(f,g)$ is a complete intersection.

There is a non empty Zariski open set $V \subset U$ such that the fiber $X(f,g)$ is a smooth complete intersection. For any pair $(f,g) \in \mathbb{P}^N \times \mathbb{P}^N$, $(f,g) \in V$ if and only if the following condition holds:

(1) The polynomial $h(t) = \det(f + tg)$ is separable over $\overline{\mathbb{Q}_p}$.
(2) For each zero λ of $h(t)$, rank $(f + \lambda g) = n$ and rank $g \geq n$, (cf. [CTSaSw], §1).

Lemma 8.2. *If every fiber $X(f,g)$, $(f,g) \in V$ has a $\mathbb{Q}_p$-rational point, every $X(f,g)$, $(f,g) \in \mathbb{P}^N \times \mathbb{P}^N$ has a $\mathbb{Q}_p$-rational point.*

Proof. Let $a \in \mathbb{P}^N \times \mathbb{P}^N$. Since $V(\mathbb{Q}_p)$ is dense in $(\mathbb{P}^N \times \mathbb{P}^N)(\mathbb{Q}_p)$, we pick $a_n \in V(\mathbb{Q}_p)$ which tends to a for the p adic topology. Let p_n be a $\mathbb{Q}_p$ point in the fiber of a_n. The points $\{p_n\}$ are contained in the compact set $Z(\mathbb{Q}_p)$ and have a convergent subsequence which converges to a point $p \in Z(\mathbb{Q}_p)$. The image of p is simply a so that the fiber at a has the $\mathbb{Q}_p$ rational point p. □

Thus using this lemma, we can construct $(\tilde{f}_i, \tilde{g}_i)$ close enough to (f_i, g_i) so that the form $\tilde{f}_i + t\tilde{g}_i$, $1 \leq i \leq 3$ has no $\mathbb{Q}_p$ point; further, $(\tilde{f}_i, \tilde{g}_i)$ define a smooth complete intersection in $\mathbb{P}^n$. In particular, $(\tilde{f}_i, \tilde{g}_i)$ satisfy the conditions 1), 2), 3) and 4).

Example 8.3. Let $f = X_1^2 - uX_2^2 - pX_3^2 + upX_4^2$, the unique anisotropic form over $\mathbb{Q}_p$, $h_1 = a_1X_5^2 + a_2X_6^2 + a_3X_7^2 + a_4X_8^2$ and $h_2 = b_1X_1^2 + b_2X_2^2 + b_3X_3^2 + b_4X_4^2$. Then for general $a_i, b_i \in \mathbb{Z}_p$, if

$$F = f(X_1, X_2, X_3, X_4) + p^4 h_1(X_5, X_6, X_7, X_8)$$

$$G = p^4 h_2(X_1, X_2, X_3, X_4) + f(X_5, X_6, X_7, X_8),$$

$F + tG$ satisfies conditions 1), 2), 3), 4). It is easy to check that $X(F,G)$ has no $\mathbb{Q}_p$ rational point. Smoothness conditions 3) and 4) are ensured by choosing general enough h_1, h_2.

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