Theta functions of lattices

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Course and project description

Let Λ be a lattice in an n-dimensional real inner product space V. We define the *theta function* $\Theta_\Lambda(z)$ of Λ as the absolutely convergent sum

$$\Theta_\Lambda(z) = \sum_{x\in\Lambda} z^{\langle x,x\rangle}$$

for $0 \le z < 1$. This is a generating function for the norms $\langle x,x\rangle$ of lattice vectors x. More generally, if $P : V \to \mathbf{C}$ is any harmonic polynomial and $\alpha : \Lambda \to \mathbf{C}$ any periodic function (a function constant on cosets of $N\Lambda$ for some $N > 0$), we define the *weighted theta function* $\Theta_{\Lambda,\alpha P}(z)$ associated to the weight αP by

$$\Theta_{\Lambda,\alpha P}(z) = \sum_{x\in\Lambda} \alpha(x)P(x)z^{\langle x,x\rangle}.$$

This generating function combines information about the norms of lattice vectors with their angular distribution and their distribution mod $N\Lambda$.

In general $z = 0$ is a branch point that prevents extension of $\Theta_{\Lambda,\alpha P}$ to a holomorphic function on the disc $|z| < 1$. But the change of variable $z = \exp(\pi i\tau)$ yields a function on the positive imaginary axis that extends to a holomorphic function on the upper half-plane. We denote this function by a lower-case θ,

$$\theta_{\Lambda,\alpha P}(\tau) = \sum_{x\in\Lambda} \alpha(x)P(x)e^{\pi i\langle x,x\rangle\tau},$$

and call it, too, a weighted theta function. Again the ordinary theta function

$$\theta_\Lambda(\tau) = \sum_{x\in\Lambda} e^{\pi i\langle x,x\rangle\tau}$$

is the special case $\alpha = P = 1$.

In number theory we are usually interested in rational lattices, that is, lattices for which the inner products are all rational: $\langle x,y\rangle \in \mathbf{Q}$ for all $x, y \in \Lambda$. Equivalently, these are lattices associated to positive-definite quadratic forms taking values in $\mathbf{Q}$. [Another equivalent condition is that there exists $m > 0$ such that $\Theta_\Lambda(z)$, and indeed every $\Theta_{\Lambda,\alpha P}(z)$, extends to a holomorphic function of $z^{1/m}$ on the unit disc for some m.] For such a lattice, $\theta_{\Lambda,\alpha P}$ is a modular form of weight $(n/2) + \deg(P)$ for some congruence subgroup of $\mathrm{PSL}_2(\mathbf{Z})$; moreover $\theta_{\Lambda,\alpha P}$ is a cusp form at least if $\deg(P) > 0$. A classical example where $n = 1$ and α and P are both nonconstant is the q-expansion of η^3:

$$\eta^3(\tau) = q^{1/8} - 3q^{9/8} + 5q^{25/8} - 7q^{49/8} - + \cdots$$

where $q = z^2 = e^{2\pi i\tau}$ as usual, and η is the modular form $q^{1/24}\prod_{k=1}^{\infty}(1 - q^k)$ of weight 1/2; here $\Lambda = \frac{1}{2}\mathbf{Z}$, $P(x) = x$, $N = 4$, and $\alpha(\frac{1}{2}n) = \chi(n)$ where χ is the

Dirichlet character of conductor 4. The "pentagonal-number" formula for $\eta(\tau)$ itself is another example, this time with P constant: $\Lambda = 12^{-1/2}\mathbf{Z}$, $P(x) = 1/2$, $N = 12$, and α is the even Dirichlet character of conductor 12.

We can then use results about modular forms, ranging from explicit information about specific spaces (e.g. a basis for the cusp forms of given weight on $\Gamma_0^+(2)$) to general properties (estimates on the growth of coefficients), to easily recover theorems on the geometry and arithmetic of rational lattices that would be much harder to approach directly.

In the AWS course, we shall review what we need of lattices, modular forms, and Poisson inversion (briefly, because that material is well known), and of harmonic polynomials (not as briefly, because though also well known this theory is not as familiar to number theorists as the other ingredients). We shall then prove that $\theta_{\Lambda,\alpha P}$ is a modular form of weight $(n/2)+\deg(P)$ when Λ is rational, and explain why this can be such a powerful tool for studying rational lattices.

We will illustrate the power of this tool with a selection of applications, emphasizing the use of forms modular for a congruence group close or equal to the full modular group $\mathrm{PSL}_2(\mathbf{Z})$. Even in this case there is a wide range of topics, such as classical formulas for counting integer solutions of $x_1^2 + \cdots + x_n^2 = N$ (the count being the $q^{N/2}$ coefficient of the modular form $\theta_{\mathbf{Z}}^n$), several cases of classification and description of lattices in $\mathbf{R}^n$ with given arithmetic invariants, and even the efficient computation of special values of L-functions. We will conclude the lectures by describing some analogues and generalizations (such as higher-genus theta functions of lattices, and weight enumerators of linear codes) and some of their uses.

The sampling of applications and variations of the modular forms $\Theta_{\Lambda,\alpha P}$ featured in the lectures will be far from exhaustive; others will provide the topics for student projects. Here is a quick overview of examples that may be treated in lectures and/or used as project topics:

- Ordinary theta series ($\alpha = P = 1$): exact or asymptotic enumeration of lattice vectors of given norm (as with the example of $\sum_{j=1}^n x_j^2 = N$ above); extremal lattices in various settings (such as even lattices with $\Lambda^* \cong c^{-1/2}\Lambda$ for $c = 1$ or $c = 2$); uniqueness of the Leech lattice and the size of its symmetry group.

- Harmonically weighted theta series ($\alpha = 1$): Asymptotic spherical equidistribution of lattice vectors of given norm $N \to \infty$ (for $n \geq 4$); spherical t-design properties of vectors of given norm in an extremal lattice, and some remarkable consequences.

- Periodically weighted theta series ($P = 1$): some uses of the theta function of the "shadow" of an odd integral lattice (one for which all inner products are integers but $\langle x, x\rangle$ takes odd values) of odd discriminant.

- Arbitrarily weighted theta series (any α, P): Asymptotic spherical equidistribution of vectors of given norm in a rational lattice translate; efficient computation of the coefficients of modular forms such as $\eta(\tau)\eta^6(2\tau)\eta^2(4\tau)$ that arise in the computation of central values of quadratic-twist families of L-functions.

- Variations and generalizations: weight enumerators of linear codes and their connections with theta functions of lattices; Siegel theta series and the distribution of r-tuples of lattice vectors with given Gram matrix.