Conjectures and open problems

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Abstract

We first consider Schanuel's Conjecture on algebraic independence of values of the exponential function. The main special case, which is yet open, is the conjecture on algebraic independence of logarithms of algebraic numbers. We survey recent work on this topic, mainly tue to D. Roy.

Next we introduce the conjecture of Kontsevich and Zagier on periods. As a special case we discuss multiple zeta values.

Finally we quote some open problems on expansions of irrational algebraic numbers.

Schanuel's Conjecture

Let $x_1, \dots, x_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Then* n *at least of the* $2n$ *numbers* $x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}$ *are algebraically independent.*

The conclusion can be phrased in terms of the transcendence degree over $\mathbf{Q}$:

$$\operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}\big(x_1, \dots, x_n,\ e^{x_1}, \dots, e^{x_n}\big) \geq n.$$

`Remark` : for almost all tuples (for Lebesgue's measure) the transcendence degree is $2n$.

Known

Lindemann–Weierstraß Theorem = case where $x_1, \ldots, x_n$ are algebraic.

Let $\beta_1, \ldots, \beta_n$ be algebraic numbers which are linearly independent over $\mathbf{Q}$. Then the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent over $\mathbf{Q}$.

Problem of Gel'fond and Schneider

Raised by A.O. Gel'fond in 1948 and Th. Schneider in 1952.

Conjecture : *if α is an algebraic number, $\alpha \neq 0$, $\alpha \neq 1$ and if β is an irrational algebraic number of degree d, then the $d-1$ numbers*

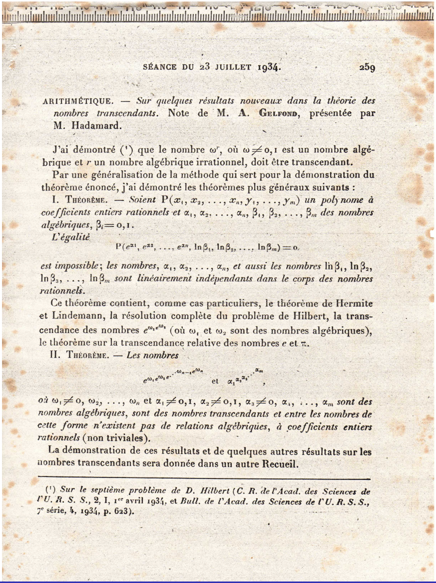
$$\alpha^{\beta},\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}$$

are algebraically independent.

Special case of Schanuel's Conjecture : take $x_i = \beta^{i-1}\log\alpha$. The conclusion is, for β algebraic number of degree d,

$$\mathrm{tr}\deg_{\mathbf{Q}}\mathbf{Q}\bigl(\log\alpha, \alpha^{\beta},\ \alpha^{\beta^2},\ \dots,\alpha^{\beta^{d-1}}\bigr) = d.$$

A.O. Gel'fond CRAS 1934



ARITHMÉTIQUE. — *Sur quelques résultats nouveaux dans la théorie des nombres transcendants.* Note de M. A. **Gelfond**, présentée par M. Hadamard.

J'ai démontré (1) que le nombre ω^r, où $\omega \neq 0, 1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les théorèmes plus généraux suivants :

I. Théorème. — *Soient* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *un polynome à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_m$ *des nombres algébriques,* $\beta_i = 0, 1$.

L'égalité

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \ln\beta_1, \ln\beta_2, \ldots, \ln\beta_m) = 0$$

est impossible; les nombres, $\alpha_1, \alpha_2, \ldots, \alpha_n$, *et aussi les nombres* $\ln\beta_1, \ln\beta_2, \ln\beta_3, \ldots, \ln\beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1 e^{\omega_2}}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. Théorème. — *Les nombres*

$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot \omega_{n-1} e^{\omega_n}}}}}} \quad \text{et} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot \alpha_m}}}}},$$

où $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ et $\alpha_1 \neq 0, 1, \alpha_2 \neq 0, 1, \alpha_3 \neq 0, \alpha_4, \ldots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels* (non triviales).

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

(1) *Sur le septième problème de D. Hilbert* (*C. R. de l'Acad. des Sciences de l'U. R. S. S.*, 2, I, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U. R. S. S.*, 7e série, 4, 1934, p. 623).

Statements by Gel'fond (1934)

Let $\beta_1, \ldots, \beta_n$ be $\mathbf{Q}$-linearly independent algebraic numbers and let $\log \alpha_1, \ldots, \log \alpha_m$ be $\mathbf{Q}$-linearly independent logarithms of algebraic numbers. Then the numbers

$$e^{\beta_1}, \ldots, e^{\beta_n}, \ \log \alpha_1, \ldots, \log \alpha_m$$

are algebraically independent over $\mathbf{Q}$.

Further statement by Gel'fond

Let $\beta_1, \dots, \beta_n$ be nonzero algebraic numbers with $\beta_1 \neq 0$ and let $\log \alpha_1, \dots, \log \alpha_m$ be logarithms of algebraic numbers with $\log \alpha_1 \neq 0$ and $\log \alpha_2 \neq 0$. Then the numbers

$$e^{\beta_1 e^{\beta_2 e^{\cdot^{\cdot^{\cdot^{\beta_{n-1} e^{\beta_n}}}}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}$$

are transcendental, and there is no nontrivial algebraic relation between such numbers.

Remark by Mathilde Herblot after the lecture : the condition on α_2 should be that it is irrational.

Roy's approach to Schanuel's Conjecture (1999)

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbf{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbf{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

Let k be a positive integer, $y_1, \ldots, y_k$ complex numbers which are linearly independent over $\mathbf{Q}$, $\alpha_1, \ldots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Roy's Conjecture equivalent to Schanuel's

Assume that, for any sufficiently large positive integer N*, there exists a non-zero polynomial* $P_N \in \mathbf{Z}[X_0, X_1]$ *with partial degree* $\le N^{t_0}$ *in* X_0*, partial degree* $\le N^{t_1}$ *in* X_1 *and height* $\le e^N$ *which satisfies*

$$\left| \left(\mathcal{D}^k P_N\right)\left(\sum_{j=1}^k m_j y_j, \prod_{j=1}^k \alpha_j^{m_j}\right) \right| \le \exp(-N^u)$$

for any non-negative integers k*,* $m_1, \ldots, m_k$ *with* $k \le N^{s_0}$ *and* $\max\{m_1, \ldots, m_k\} \le N^{s_1}$*. Then*

$$\operatorname{tr}\deg \mathbf{Q}(y_1, \ldots, y_k, \alpha_1, \ldots, \alpha_k) \ge k.$$

Equivalence between Schanuel and Roy

Let $(y,\alpha)\in \mathbf{C}\times\mathbf{C}^\times$, and let s_0,s_1,t_0,t_1,u be positive real numbers satisfying the inequalities of Roy's Conjecture.
Then the following conditions are equivalent :
(a) The number αe^{-y} is a root of unity.

(b) For any sufficiently large positive integer N, there exists a nonzero polynomial $Q_N\in\mathbf{Z}[X_0,X_1]$ with partial degree $\le N^{t_0}$ in X_0, partial degree $\le N^{t_1}$ in X_1 and height $\mathrm{H}(Q_N)\le e^N$ such that

$$\bigl|(\mathcal{D}^k Q_N)(my,\alpha^m)\bigr|\le \exp(-N^u)$$

for any $k,m\in\mathbf{N}$ with $k\le N^{s_0}$ and $m\le N^{s_1}$.

Conjecture of algebraic independence of logarithms of algebraic numbers

The most important special case of Schanuel's Conjecture is

Conjecture. *Let* $\lambda_1, \ldots, \lambda_n$ *be* $\mathbf{Q}$*-linearly independent complex numbers. Assume that the numbers* $e^{\lambda_1}, \ldots, e^{\lambda_n}$ *are algebraic. Then the numbers* $\lambda_1, \ldots, \lambda_n$ *are algebraically independent over* $\mathbf{Q}$*.*

Not yet known that the transcendence degree is ≥ 2.

Reformulation by D. Roy

Instead of taking logarithms of algebraic numbers and looking for the algebraic independence relations, D. Roy fixes a polynomial and looks at the points, with coordinates logarithms of algebraic numbers, on the corresponding hypersurface.

Denote by $\mathcal{L}$ the set of complex numbers λ for which e^λ is algebraic. Hence $\mathcal{L}$ is a $\mathbf{Q}$-vector subspace of $\mathbf{C}$. Roy's statement is :

Conjecture. *For any algebraic subvariety V of $\mathbf{C}^n$ defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers, the set $V \cap \mathcal{L}^n$ is the union of the sets $E \cap \mathcal{L}^n$, where E ranges over the set of vector subspaces of $\mathbf{C}^n$ which are contained in V.*

Algebraic independence and simultaneous approximation

Let $\underline{\theta} = (\theta_1, \ldots, \theta_m)$ be a tuple of complex numbers such that the number

$$t = \operatorname{tr}\deg_{\mathbf{Q}} \mathbf{Q}(\underline{\theta})$$

is ≥ 1. There exist two positive constants c_1 and c_2 with the following property. Let $(D_\nu)_{\nu \geq 1}$ and $(\mu_\nu)_{\nu \geq 1}$ be sequences of real numbers satisfying $D_\nu \geq c_1$, $\mu_\nu \geq c_1$,

$$c_1 \leq D_\nu \leq D_{\nu+1} \leq 2D_\nu,$$

and

$$c_1 D_\nu \leq \mu_\nu \leq \mu_{\nu+1} \leq 2\mu_\nu \qquad (\nu \geq 1).$$

Algebraic independence and simultaneous approximation

Assume also

$$\lim_{\nu\to\infty} \mu_\nu = \infty.$$

Then for infinitely many ν *there exists a* m*-tuple* $\underline{\gamma} = (\gamma_1, \dots, \gamma_m)$ *of algebraic numbers satisfying*

$$[\mathbf{Q}(\underline{\gamma}) : \mathbf{Q}] \le D_\nu, \quad [\mathbf{Q}(\underline{\gamma}) : \mathbf{Q}] \max_{1\le i\le m} \mathrm{h}(\gamma_i) \le \mu_\nu$$

and

$$\max_{1\le i\le m} |\theta_i - \gamma_i| \le e^{-c_2 D_\nu^{1/t} \mu_\nu}.$$

Measure of simultaneous approximation

Conjecture *There exist two positive absolute constants* c_1 *and* c_2 *with the following property. Let* $\lambda_1, \ldots, \lambda_m$ *be logarithms of algebraic numbers with* $\alpha_i = e^{\lambda_i} \quad (1 \le i \le m)$, *let* $\beta_0, \ldots, \beta_m$ *be algebraic numbers,* D *the degree of the number field* $\mathbf{Q}(\alpha_1, \ldots, \alpha_m, \beta_0, \ldots, \beta_m)$ *and finally let* $h \ge 1/D$ *satisfy*

$$h \ge \max_{1 \le i \le m} h(\alpha_i), \quad h \ge \frac{1}{D} \max_{1 \le i \le m} |\lambda_i| \quad \textit{and} \quad h \ge \max_{0 \le j \le m} h(\beta_j).$$

Assume $\lambda_1, \ldots, \lambda_m$ *are linearly independent over* $\mathbf{Q}$. *Then*

$$\sum_{i=1}^{m} |\lambda_i - \beta_i| \ge \exp\bigl\{-c_2 m D^{1+(1/m)} h\bigr\}.$$

Structural rank of a matrix

Let K be a field, k a subfield and $\mathtt{M}$ a matrix with entries in K. Following D. Roy, we define *structural rank of* $\mathtt{M}$ *with respect to* k

Consider the k-vector subspace $\mathcal{E}$ of K spanned by the entries of $\mathtt{M}$. Choose an injective morphism φ of $\mathcal{E}$ into a k-vector space $kX_1 + \cdots + kX_n$. The image $\varphi(\mathtt{M})$ of $\mathtt{M}$ is a matrix whose entries are in the field $k(X_1, \ldots, X_n)$ of rational fractions. Its rank does not depend on the choice of φ.

This is the structural rank of $\mathtt{M}$ with respect to k.

Homogeneous algebraic independence of logarithms

According to D. Roy, the homogeneous case of the conjecture on algebraic independence of logarithms of algebraic numbers is equivalent to :

Conjecture. *Let* $\mathtt{M}$ *be a matrix whose entries are logarithms of algebraic numbers. Then the rank of* $\mathtt{M}$ *is equal to its structural rank with respect to* $\mathbf{Q}$.

Algebraic independence of logarithms

According to D. Roy, the conjecture on algebraic independence of logarithms of algebraic numbers is equivalent to :

Conjecture. *Any matrix*

$$\bigl(b_{ij} + \lambda_{ij}\bigr)_{\substack{1\le i\le d\\ 1\le j\le \ell}}$$

with $b_{ij} \in \mathbf{Q}$ *and* $\lambda_{ij} \in \mathcal{L}$ *has a rank equal to its structural rank.*

Any Polynomial is the Determinant of a Matrix

The proof of the equivalence uses the nice auxiliary result :

For any $P \in k[X_1, \ldots, X_n]$ *there exists a square matrix with entries in the* k*-vector space* $k + kX_1 + \cdots + kX_n$ *whose determinant is* P.

Partial result

D. Roy proved the following extension of the *Strong six exponentials Theorem.*
the rank of a matrix whose entries are logarithms of algebraic numbers is at least half its structural rank with respect to $\mathbf{Q}$

and also that

the rank of a matrix whose entries are linear combinations of logarithms of algebraic numbers with algebraic coefficients is at least half its structural rank with respect to the field $\overline{\mathbb{Q}}$ *of algebraic numbers.*

The Strong Four Exponentials Conjecture

Denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbb{Q}}$-vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers :

$$\widetilde{\mathcal{L}} = \{\beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n \; ; \; n \geq 0, \beta_i \in \overline{\mathbb{Q}}, \; \lambda_i \in \mathcal{L}\}.$$

Assume the strong Four Exponentials Conjecture.

- If Λ is in $\widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$ then the quotient $1/\Lambda$ is not in $\widetilde{\mathcal{L}}$.
- If Λ_1 and Λ_2 are in $\widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$, then the product $\Lambda_1\Lambda_2$ is not in $\widetilde{\mathcal{L}}$.
- If Λ_1 and Λ_2 are in $\widetilde{\mathcal{L}}$ with Λ_1 and Λ_2/Λ_1 transcendental, then this quotient Λ_2/Λ_1 is not in $\widetilde{\mathcal{L}}$.

Transcendence of e^{π^2}

- Open problem : *is the number* e^{π^2} *transcendental ?*

- More generally : *for* $\lambda \in \mathcal{L} \setminus \{0\}$, *is it true that* $\lambda\overline{\lambda} \notin \mathcal{L}$ *?*

- More generally : *for* λ_1 *and* λ_2 *in* $\mathcal{L} \setminus \{0\}$, *is it true that* $\lambda_1\lambda_2 \notin \mathcal{L}$ *?*

- *For* λ_1 *and* λ_2 *in* $\mathcal{L} \setminus \{0\}$, *is it true that* $\lambda_1\lambda_2 \notin \widetilde{\mathcal{L}}$ *?*

A conjecture of G. Diaz

Diaz' Conjecture. *Let* $u \in \mathbf{C}^\times$. *Assume* $|u|$ *is algebraic. Then* e^u *is transcendental.*

- G. Diaz – « Utilisation de la conjugaison complexe dans l'étude de la transcendance de valeurs de la fonction exponentielle », *J. Théor. Nombres Bordeaux* **16** (2004), p. 535–553.
- G. Diaz – « Produits et quotients de combinaisons linéaires de logarithmes de nombres algébriques : conjectures et résultats partiels », Submitted (2005), 19 p.

Mahler's problem (1967)

- For a and b positive integers,

$$|e^b - a| > a^{-c}?$$

- **Stronger conjecture** :

$$|e^b - a| > b^{-c}?$$

- K. Mahler (1953, 1967), M. Mignotte (1974), F. Wielonsky (1997) :

$$|e^b - a| > b^{-20b}$$

- Joint work with Yu.V. Nesterenko (1996) for a and b rational numbers, refinement by S. Khemira and P. Voutier.

Exact rounding of the elementary functions

Applications in theoretical computer science :

Muller, J-M. ; Tisserand, A. –

Towards exact rounding of the elementary functions. Alefeld, Goetz (ed.) et al.,

Scientific computing and validated numerics.

Proceedings of the international symposium on scientific computing, computer arithmetic and validated numerics SCAN-95, Wuppertal, Germany, September 26-29, 1995.

Berlin : Akademie Verlag. Math. Res. 90, 59-71 (1996).

p–adic transcendental numbers

Two open problems : the radius of convergence of $\exp_p$ is finite :

$$n! \to \infty \text{ but } |n!|_p \to 0$$

Problem 1 : *p–adic analogue of the Lindemann – Weierstraß' Theorem*

Problem 2 : *p–adic analogue of Gel'fond's Theorem on the algebraic independence of α^β and α^{β^2} for β cubic irrational.*

W.W. Adams (1966) :
For $[\mathbf{Q}(\beta):\mathbf{Q}] = d \geq 4$, two of the numbers α^β, α^{β^2} , $\ldots\alpha^{\beta^{d-1}}$ are algebraically independent.
For $[\mathbf{Q}(\beta):\mathbf{Q}] = 3$, two of the numbers $\log\alpha$, α^β and α^{β^2} are algebraically independent.

Periods : Maxime Kontsevich and Don Zagier

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational fractions with rational coefficients on subsets of $\mathbf{R}^n$ defined by (in)equalities defined by rational polynomials.

Periods, *Mathematics unlimited—2001 and beyond*, Springer 2001, 771–808.

Basic examples

Basic example of a *period* :

$$e^{z+2i\pi} = e^z$$

$$\pi = \int_{x^2+y^2\leq 1} dxdy,$$

Periods and quasi periods of an elliptic curve, elliptic and abelian integrals

$$\omega_i = \int_{e_i}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad (i = 1, 2)$$

where

$$4t^3 - g_2 t - g_3 = 4(t - e_1)(t - e_2)(t - e_3).$$

Elliptic integrals

$$\int_1^\infty \frac{dt}{\sqrt{t^3-t}} = \frac{1}{2}B(1/4,1/2) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}}$$

and

$$\int_1^\infty \frac{dt}{\sqrt{t^3-1}} = \frac{1}{3}B(1/6,1/2) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi}$$

$$\sqrt{2}\int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{1}{2^{3/2}}B(1/4,1/2) = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}$$

$$2\int_{-b}^{b}\sqrt{1+\frac{a^2x^2}{b^4-b^2x^2}}\;dx$$

Examples of a non–period

Open problem : *Give an explicit example of a complex number which is not a period.*

Several levels :

- *analog of Liouville* : produce a property which is satisfied by all periods and construct a number which does not share this property.
  Suggestion : in terms of complexity ?
- *analog of Hermite* : prove that some specific numbers are not periods.
  Candidates : $1/\pi$, e, Euler constant
  M. Kontsevich : exponential periods.
  The last chapter, which is at a more advanced level and also more speculative than the rest of the text, is by the first author only.

Examples of periods

$$\sqrt{2} = \int_{2x^2 \leq 1} dx$$

as well as any algebraic number.

$$\log 2 = \int_{1<x<2} \frac{dx}{x}$$

as well as any logarithm of an algebraic number.

$$\pi = \int_{x^2+y^2 \leq 1} dxdy,$$

$$\frac{\pi^2}{6} = \zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2}.$$

$\zeta(2)$ is a period

$$\begin{aligned}\int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \cdot \frac{dt_2}{1-t_2} &= \int_0^1 \left(\int_0^{t_1} \frac{dt_2}{1-t_2} \right) \frac{dt_1}{t_1} \\ &= \int_0^1 \left(\int_0^{t_1} \sum_{n\geq 1} t_2^{n-1} dt_2 \right) \frac{dt_1}{t_1} \\ &= \sum_{n\geq 1} \frac{1}{n} \int_0^1 t_1^{n-1} dt_1 \\ &= \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2).\end{aligned}$$

$\zeta(s)$ is a period

For s a positive integer ≥ 2,

$$\zeta(s) = \int_{1>t_1>t_2\cdots>t_s>0} \frac{dt_1}{t_1} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s}.$$

Induction :

$$\int_{t_1>t_2\cdots>t_s>0} \frac{dt_2}{t_2} \cdots \frac{dt_{s-1}}{t_{s-1}} \cdot \frac{dt_s}{1-t_s} = \sum_{n\geq 1} \frac{t_1^{n-1}}{n^{s-1}}.$$

Relations between periods

1

Additivity

$$\int_a^b \bigl(f(x)+g(x)\bigr)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

et

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

2

Change of variables

$$\int_{\varphi(a)}^{\varphi(b)} f(t)dt = \int_a^b f\bigl(\varphi(u)\bigr)\varphi'(u)du.$$

Relations between periods

3

Newton–Leibniz–Stokes

$$\int_a^b f'(t)dt = f(b) - f(a).$$

Conjecture of Kontsevich and Zagier

Periods,
*Mathematics unlimited—
2001 and beyond,*
Springer 2001, 771–808.

Conjecture (Kontsevich–Zagier). *If a period has two representations, they can be deduced one from the other using only rules* [1], [2] *and* [3] *in which all functions and integration domains are algebraic with algebraic coefficients.*

Examples

$$\begin{aligned}
\pi &= \int_{x^2+y^2\le 1} dxdy & &= 2\int_{-1}^{1}\sqrt{1-x^2}\,dx\\
&= \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} & &= \int_{-\infty}^{\infty}\frac{dx}{1+x^2}\\
&= \frac{22}{7}-\int_0^1\frac{x^4(1-x^4)dx}{1+x^2} & &= 4\int_0^1\frac{dx}{1+x^2}.
\end{aligned}$$

Dramatic consequences

No new algebraic dependence relation among classical constants from analysis.

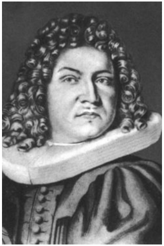
Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

Euler : $s \in \mathbf{R}$.

Riemann : $s \in \mathbf{C}$.

Special values of Riemann zeta function



$s \in \mathbf{Z}$:
Jacques Bernoulli (1654–1705),
Leonard Euler (1739).

$\pi^{-2k}\zeta(2k) \in \mathbf{Q}$ *for* $k \geq 1$ (Bernoulli numbers).

Diophantine question

Describe all algebraic relations among

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \ldots$$

Conjecture. *The numbers*

$$\zeta(2), \quad \zeta(3), \quad \zeta(5), \quad \zeta(7), \ldots$$

are algebraically independent.

Linearization of the situation *(Euler)*

The product of two special values of the zeta function is a sum of *multizeta* values.

$$\sum_{n_1\geq 1} n_1^{-s_1} \sum_{n_2\geq 1} n_2^{-s_2} = \sum_{n_1>n_2\geq 1} n_1^{-s_1} n_2^{-s_2} + \sum_{n_2>n_1\geq 1} n_2^{-s_2} n_1^{-s_1} + \sum_{n\geq 1} n^{-s_1-s_2}$$

Linearization of the situation *(Euler)*

For $s_1 \geq 2$ and $s_2 \geq 2$, we have

$$\zeta(s_1)\zeta(s_2) = \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2)$$

with

$$\zeta(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} n_1^{-s_1} n_2^{-s_2}.$$

Multizeta values

For k, $s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$, set $\underline{s} = (s_1, \ldots, s_k)$ et

$$\zeta(\underline{s}) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}}.$$

For $k = 1$ these are special values of the ζ function.

k is the *depth* and $p = s_1 + \cdots + s_k$ the *weight*.

The MZV algebra

The product of two multizeta values is a multizeta value.

Hence the $\mathbf{Q}$–vector space spanned by the numbers $\zeta(\underline{s})$ is also a $\mathbf{Q}$–algebra.

The problem of algebraic independence is reduced to a question of linear independence.

Question : which are the linear relations among these numbers ?

Answer : *there are many linear relations !*

Zagier's Conjecture

Denote by $\mathfrak{Z}_p$ the $\mathbf{Q}$-subspace of $\mathbf{R}$ spanned by the real numbers $\zeta(\underline{s})$ where $\underline{s}$ has weight $s_1 + \cdots + s_k = p$, with $\mathfrak{Z}_0 = \mathbf{Q}$ and $\mathfrak{Z}_1 = \{0\}$.
Let d_p denote the dimension of $\mathfrak{Z}_p$.

Conjecture (Zagier). *For $p \geq 3$, we have*

$$d_p = d_{p-2} + d_{p-3}.$$

$$(d_0,\ d_1,\ d_2, \ldots) = (1, 0, 1, 1, 1, 2, 2, \ldots).$$

Hoffman's conjecture

Zagier's Conjecture can be written

$$\sum_{p\geq 0} d_p X^p = \frac{1}{1-X^2-X^3}.$$

M. Hoffman Conjecture : *a basis of* $\mathfrak{Z}_p$ *as a* $\mathbf{Q}$*–vector space is given by the numbers* $\zeta(s_1,\dots,s_k)$, $s_1+\cdots+s_k=p$*, where each* s_i *is* 2 *or* 3*.*

True for $p\leq 20$:

M. Kaneko, M. Noro and K. Tsurumaki. – *On a conjecture for the dimension of the space of the multiple zeta values*, Software for Algebraic Geometry, IMA **148** (2008), 47–58.

Upper bound for the dimension

A.B. Goncharov – *Multiple ζ-values, Galois groups and Geometry of Modular Varieties.* Birkhäuser. Prog. Math. **201**, 361-392 (2001).
T. Terasoma – *Mixed Tate motives and Multiple Zeta Values.* Invent. Math. **149**, No.2, 339-369 (2002).

Theorem. *The numbers given by Zagier's conjecture $d_p = d_{p-2} + d_{p-3}$ with initial conditions $d_0 = 1$, $d_1 = 0$ are upper bounds for the dimension of $\mathfrak{Z}_p$.*

Émile Borel

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  http ://www.emis.de/MATH/JFM/JFM.html
- *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Decimal expansion of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.414213562373095048801688724209698078569671875376948073176679737990732478462107038850387534327641572735013846230912297024924836055850737212644121497099935831413222665927505592755799950501152782060571470109559971605970274534596862014728517418640889198609552329230484308714321450839762603627995251407989687253396546331808829640620615258352395054745750287759961729835575220337531857011354374603408498847160386899970699004815030544027790316454247823068492936918621580578463111596668713013015618568987237235288509264861249497715421833420428568606014682472077143585487415565706967765372022648544701585880162075847492265722600208558446652145839889394437092659180031138824646815708263010059485870400318648034219489727829064104507263688131373985525611732204024509122770022694112757362728049573810896750401836986836845072579936472906076299694138047565482372899718032680247442062926912485905218100445984215059112024944134172853147810580360337107730918286931471017111168391658172688941975871658215212822951848847 . . .

Binary expansion of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.0110101000001001111001100110011111110011101111001100100100000
1000101100101111101100010011011001101110101010010101011111010100
1111100011101011011110110000010111010100010010011101110101010000
1001100111011010001011110101100100000101100000011001100111001100
1000101010100101011111100100000110000010000111010101011100010100
0101100001110101000101100011111111001101111110111001000001111 0
1101100111001000011110111010010101000010111100100001110011100 0
1111011010010100111100000000010010000111001101100011110111111 01
0001001110110100011010010001000000001011101000011101000010101 01
1110001111101001110010100110000010110011100011000000000100011 01
1110000110011011110111100101010110001101111001001000100010110 1
0001000010001011000101001000110000010101011110001110010001011 1
1011111000100111000110011110001101101010110101000101000111000 1
0111011011111101001110111001100101100101010011000110100001100 1
1000111110011110010000100110111111010100101111000100100000111 11
0000011011011100101100000010111011101010101001001010000001000100
1100100000010000000110010100100101010000000100111001010010 10 . . .

Expansion in basis g of a real algebraic number

Let $g \geq 2$ be an integer and x a real algebraic irrational number.

• É. Borel : *The expansion in basis g of x should obey to some of the laws which are shared by almost all numbers for Lebesgue's measure.*

• **Remark** : no number could obey **all** laws which are shared by all numbers outside a set of measure zero because the intersection of these sets of measure 1 is empty !

$$\bigcap_{x \in \mathbf{R}} \mathbf{R} \setminus \{x\} = \emptyset.$$

• Precise statements by Y. Bugeaud and B. Adamczewski.

Suggestion by Émile Borel

• In the basis g expansion of a real algebraic irrational number, *each of the digits* $0, 1, \ldots, g-1$ *should occur at least once.*

• As a consequence, one would deduce that *any given sequence of digits should occur infinitely often in the expansion of any irrational algebraic real number.*

• `Hint` : replace g by a power of g.

Normal numbers, according to Borel

Let g be an integer with $g \geq 2$.

• A real number is *simply normal in basis* g if any of the digits $\{0, 1, \ldots, g-1\}$ occurs in its expansion in basis g with frequency $1/g$.

• A real number x is *normal in basis* g if x is simply normal in basis g^n for all $n \geq 2$.

• A real number is normal if x is normal in any basis $g \geq 2$.

• *Almost all numbers are normal* (É. Borel, 1909).

Borel's suggestion vs the state of the art

• Borel suggested that *any real irrational algebraic number is normal.*

• *There is no explicitly known triple* (g, a, x), *where* $g \geq 3$ *is an integer,* a *a digit in* $\{0, \ldots, g-1\}$ *and* x *a real irrational algebraic number, for which one could tell whether the digit* a *occurs infinitely often in the expansion of* x *in basis* g.

Automatic sequences

Theorem (B. Adamczewski, Y. Bugeaud, F. Luca, 2004 –conjecture of A. Cobham, 1968) : *The sequence of digits in a basis* $g \geq 2$ *of an irrational algebraic number is not automatic.*

Continued fractions

Siimilar questions occur for the continued fracion expansion of algebraic real numbers of degree ≥ 3.

Open question – A.Ya. Khintchine (1949) : *are the partial quotients of the continued fraction expansion of a non–quadratic irrational algebraic real number bounded ?*

No example is known :

• We do not know whether there exists a non–quadratic irrational algebraic real number with bounded partial quotients in its continued fraction expansion.

• We do not know whether there exists an irrational algebraic real number with unbounded partial quotients in its continued fraction expansion.

Conjectures and open problems

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