Transcendence in Positive Characteristic

t-Motives and Difference Galois Groups

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Outline

1. t-Motives
2. Difference Galois groups
3. Algebraic independence

t-Motives

- Definitions
- Connections with Drinfeld modules and t-modules
- Rigid analytic triviality

Scalar quantities

Let p be a fixed prime; q a fixed power of p.

$$A := \mathbb{F}_q[\theta] \quad \longleftrightarrow \quad \mathbb{Z}$$

$$k := \mathbb{F}_q(\theta) \quad \longleftrightarrow \quad \mathbb{Q}$$

$$\overline{k} \quad \longleftrightarrow \quad \overline{\mathbb{Q}}$$

$$k_\infty := \mathbb{F}_q((1/\theta)) \quad \longleftrightarrow \quad \mathbb{R}$$

$$\mathbb{C}_\infty := \widehat{\overline{k_\infty}} \quad \longleftrightarrow \quad \mathbb{C}$$

$$|f|_\infty = q^{\deg f} \quad \longleftrightarrow \quad |\cdot|$$

Functions

- Rational functions:

$$\mathbb{F}_q(t), \quad \overline{k}(t), \quad \mathbb{C}_\infty(t).$$

- Analytic functions:

$$\mathbb{T} := \left\{ \sum_{i \geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; |a_i|_\infty \to 0 \right\}.$$

and

$$\mathbb{L} := \text{fraction field of } \mathbb{T}.$$

- Entire functions:

$$\mathbb{E} := \left\{ \sum_{i \geq 0} a_i t^i \in \mathbb{C}_\infty[[t]] \;\middle|\; \begin{array}{c} \sqrt[i]{|a_i|_\infty} \to 0, \\ [k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty \end{array} \right\}.$$

The ring $\overline{k}[t, \boldsymbol{\sigma}]$

The ring $\overline{k}[t, \boldsymbol{\sigma}]$ is the non-commutative polynomial ring in t and $\boldsymbol{\sigma}$ with coefficients in $\overline{k}$, subject to

$$tc = ct, \quad t\boldsymbol{\sigma} = \boldsymbol{\sigma} t, \quad \boldsymbol{\sigma} c = c^{1/q}\boldsymbol{\sigma}, \quad \forall\, c \in \overline{k}.$$

Thus for any $f \in \overline{k}[t]$,

$$\boldsymbol{\sigma} f = f^{(-1)}\boldsymbol{\sigma} = \sigma(f)\boldsymbol{\sigma}.$$

Anderson t-motives

Definition

An *Anderson t-motive* M is a left $\overline{k}[t, \sigma]$-module such that

- M is free and finitely generated over $\overline{k}[t]$;
- M is free and finitely generated over $\overline{k}[\sigma]$;
- $(t-\theta)^n \mathrm{M} \subseteq \sigma \mathrm{M}$ for $n \gg 0$.

Anderson t-motives form a category in which morphisms are simply morphisms of left $\overline{k}[t, \sigma]$-modules.

Connections with Drinfeld modules

Theorem (Anderson 1986)

The category of Anderson t-motives contains the categories of Drinfeld modules and (abelian) t-modules over $\overline{k}$ as full subcategories.

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Suppose M is an Anderson t-Motive that corresponds to a Drinfeld module (or t-module) $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$. How do we recover ρ from M?

$$\rho(\overline{k}) \cong \frac{\mathrm{M}}{(\boldsymbol{\sigma} - 1)\mathrm{M}}.$$

The Carlitz motive

Let $\mathsf{C} = \overline{k}[t]$ and define a left $\overline{k}[\sigma]$-module structure on C by setting

$$\sigma(f) = (t - \theta)f^{(-1)}, \quad \forall f \in \mathsf{C}.$$

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$$\boldsymbol{\sigma}(f) = (t-\theta)f^{(-1)}, \quad \forall f \in \mathsf{C}.$$

For $x \in \overline{k}$, we see that

$$\begin{aligned} tx = \theta x + (t-\theta)x &= \theta x + \boldsymbol{\sigma}(x^q) \\ &= \theta x + x^q + (\boldsymbol{\sigma} - 1)(x^q) \\ &= C(t)(x) + (\boldsymbol{\sigma} - 1)(x^q). \end{aligned}$$

So as $\mathbb{F}_q[t]$-modules,

$$\text{Carlitz module} \cong \frac{\mathsf{C}}{(\boldsymbol{\sigma} - 1)\mathsf{C}}.$$

Representations of σ

Suppose M is an Anderson t-motive and that $m_1, \ldots, m_r \in$ M form a $\overline{k}[t]$-basis of M. Let

$$\mathbf{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix}.$$

Then we can define $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ by

$$\sigma\mathbf{m} = \begin{bmatrix} \sigma m_1 \\ \vdots \\ \sigma m_r \end{bmatrix} = \Phi \begin{bmatrix} m_1 \\ \vdots \\ m_r \end{bmatrix}.$$

We say that Φ *represents multiplication by* σ on M.

t-Motives for rank 2 Drinfeld modules

Suppose that $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ is a rank 2 Drinfeld module with

$$\rho(t) = \theta + \kappa F + F^2.$$

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Suppose that $\mathrm{M} = \mathrm{Mat}_{1\times 2}(\overline{k}[t])$ is the Anderson t-motive with multiplication by σ represented by

$$\Phi = \begin{bmatrix} 0 & 1 \\ t - \theta & -\kappa^{1/q} \end{bmatrix}.$$

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Suppose that $\mathsf{M} = \mathrm{Mat}_{1\times 2}(\overline{k}[t])$ is the Anderson t-motive with multiplication by $\boldsymbol{\sigma}$ represented by

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Then

$$\rho \cong \frac{\mathsf{M}}{(\boldsymbol{\sigma} - 1)\mathsf{M}}.$$

Indeed,

$$
\begin{aligned}
t[x,0] = [tx,0] &= [tx + \kappa x^q, -\kappa^{(-1)}x] + [-\kappa x^q, \kappa^{(-1)}x] \\
&= [tx + \kappa x^q, -\kappa^{1/q}x] + (\boldsymbol{\sigma} - 1)[\kappa x^q, 0] \\
&= [\theta x + \kappa x^q + x^{q^2}, 0] + [(t-\theta)x - x^{q^2}, -\kappa^{1/q}x] \\
&\qquad + (\boldsymbol{\sigma} - 1)[\kappa x^q, 0] \\
&= [\theta x + \kappa x^q + x^{q^2}, 0] \\
&\qquad + (\boldsymbol{\sigma} - 1)[\kappa x^q, 0] + (\boldsymbol{\sigma}^2 - 1)[x^{q^2}, 0].
\end{aligned}
$$

Rigid analytic triviality

In the examples we have seen, we have the following chain of constructions:

$$\left\{ \begin{array}{l} \text{Drinfeld module} \\ \text{or } t\text{-module } \rho \end{array} \right\} \Longrightarrow \left\{ t\text{-motive M} \right\}$$
$$\Longrightarrow \left\{ \begin{array}{l} \Phi \in \mathrm{Mat}_r(\bar{k}[t]) \\ \text{representing } \boldsymbol{\sigma} \end{array} \right\}$$
$$\stackrel{(\star)}{\Longrightarrow} \left\{ \begin{array}{l} \Psi \in \mathrm{Mat}_r(\mathbb{E}), \\ \Psi^{(-1)} = \Phi\Psi \end{array} \right\}$$
$$\Longrightarrow \left\{ \begin{array}{l} \Psi(\theta)^{-1} \text{ provides} \\ \text{periods of } \rho \end{array} \right\}.$$

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$$\Longrightarrow \left\{\begin{matrix}\Psi(\theta)^{-1} \text{ provides} \\ \text{periods of } \rho\end{matrix}\right\}.$$

Everything goes through fine, as long as we can do $(\star)$.

Rigid analytic triviality

Definition

An Anderson t-motive M is *rigid analytically trivial* if for $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ representing multiplication by σ on M, there exists a (fundamental matrix)

$$\Psi \in \mathrm{Mat}_r(\mathbb{E}) \cap \mathrm{GL}_r(\mathbb{T})$$

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A deep theorem of Anderson proves the following equivalence,

$$\left\{\begin{array}{l}\text{Drinfeld module or } t\text{-} \\ \text{module is uniformizable}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} t\text{-motive M is rigid} \\ \text{analytically trivial}\end{array}\right\}.$$

Difference Galois groups

- Definitions and constructions
- Properties
- Connections with t-motives/Drinfeld modules

Preliminaries

We will work in some generality. We fix fields $K \subseteq L$ with an automorphism $\sigma : L \xrightarrow{\sim} L$ such that

- $\sigma(K) \subseteq K$;
- L/K is separable;
- $L^\sigma = K^\sigma =: E$.

The example to keep in mind of course is $(E, K, L) = (\mathbb{F}_q(t), \overline{k}(t), \mathbb{L})$.

Σ and Λ

We suppose that we have matrices $\Phi \in \mathrm{GL}_r(K)$ and $\Psi \in \mathrm{GL}_r(L)$ so that

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Let $\Sigma = \operatorname{im} \nu$ and take Λ for its fraction field in L:

$$\Sigma = K[\Psi, 1/\det \Psi], \quad \Lambda = K(\Psi).$$

Additional hypothesis: K is algebraically closed in Λ.

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Let $\Sigma = \operatorname{im} \nu$ and take Λ for its fraction field in L:

$$\Sigma = K[\Psi, 1/\det \Psi], \quad \Lambda = K(\Psi).$$

Additional hypothesis: K is algebraically closed in Λ. Generally holds in the case $(\mathbb{F}_q(t), \overline{k}(t), \mathbb{L})$.

The Galois group Γ

Let $Z \subseteq \mathrm{GL}_{r/K}$ be the smallest K-subscheme such that $\Psi \in Z(L)$. Thus,

$$Z \cong \operatorname{Spec} \Sigma \quad \text{(as K-schemes).}$$

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Now set $\Psi_1, \Psi_2 \in \mathrm{GL}_r(L \otimes_K L)$ so that

$$(\Psi_1)_{ij} = \Psi_{ij} \otimes 1, \quad (\Psi_2)_{ij} = 1 \otimes \Psi_{ij},$$

and set $\widetilde{\Psi} = \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(L \otimes_K L)$.

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and set $\widetilde{\Psi} = \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(L \otimes_K L)$.
Define an E-algebra map,

$$\mu = (X_{ij} \mapsto \widetilde{\Psi}_{ij}) : E[X, 1/\det X] \to L \otimes_K L,$$

which defines a closed E-subscheme Γ of $\mathrm{GL}_{r/E}$.

Working hypotheses for Γ and Z

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$$(\alpha, \beta) \mapsto (\alpha, \alpha\beta) : Z \times \Gamma \xrightarrow{\sim} Z \times Z.$$

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- Everything should be done in such a way as to be defined over the smallest field possible (say E, K, or L).

The difference Galois group Γ

Theorem (P. 2008)

- *Γ is a closed E-subgroup scheme of $\mathrm{GL}_{r/E}$.*
- *Z is stable under right-multiplication by Γ_K and is a Γ_K-torsor.*
- *The K-scheme Z is absolutely irreducible and is smooth over $\overline{K}$.*
- *The E-scheme Γ is absolutely irreducible and is smooth over $\overline{E}$.*
- *The dimension of Γ over E is equal to the transcendence degree of Λ over K.*
- $\Gamma(E) \cong \mathrm{Aut}_\sigma(\Lambda/K)$.
- *If every element of $\overline{E}$ is fixed by some power of σ, then the elements of Λ fixed by $\Gamma(\overline{E})$ are precisely K.*

Connections with t-motives

Given a rigid analytically trivial Anderson t-motive M, we form

$$M := \overline{k}(t) \otimes_{\overline{k}[t]} \mathsf{M}.$$

Then M carries the structure of a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module with

- M is a f.d. $\overline{k}(t)$-vector space;
- multiplication by σ on M is represented by a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ that has a fundamental matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$.

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Proposition (P. 2008)

The objects just described form a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor

$$\omega(M) = (\mathbb{L} \otimes_{\overline{k}(t)} M)^{\sigma}.$$

Category of t-motives

- "neutral Tannakian category over $\mathbb{F}_q(t)$" $\Longleftrightarrow$ category of representations of an affine group scheme over $\mathbb{F}_q(t)$.
- We define the *category of t-motives* to be the Tannakian subcategory generated by all Anderson t-motives.

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Theorem (P. 2008)

Let M be a t-motive. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by σ on M and that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization for Φ. Then the Galois group Γ_Ψ associated to the difference equations

$$\Psi^{(-1)} = \Phi\Psi$$

is naturally isomorphic to the group Γ_M associated to M via Tannakian duality.

Algebraic independence

- Main theorem
- Sketch of the proof

Galois groups and transcendence

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Let M be a t-motive, and let Γ_M be its associated group via Tannakian duality. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^{\times}$.

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$$L = \overline{k}(\Psi(\theta)) \subseteq \overline{k_\infty}.$$

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$$\mathrm{tr.\,deg}_{\overline{k}}\, L = \dim \Gamma_M.$$

Remarks: If M arises from an actual Anderson t-motive, then the hypotheses of the theorem are automatically satisfied.

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In practice to calculate $\dim \Gamma_M$, we calculate Γ_Ψ.

Sketch of the proof

Needless to say the proof relies heavily on the ABP-criterion.

- Fix $d \geq 1$. For each $n \geq 1$, the entries of the Kronecker product,

$$\Psi^{\otimes n},$$

  are all degree n monomials in the entries of Ψ.

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- Let $\overline{\psi}$ be the column vector whose entries are the concatenation of 1 and each of the columns of $\Psi^{\otimes n}$ for $n \leq d$. Let
$$\overline{\Phi} = [1] \oplus \Phi^{\oplus r} \oplus (\Phi^{\otimes 2})^{\oplus r^2} \oplus \cdots \oplus (\Phi^{\otimes d})^{\oplus r^d}.$$

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- Then

$$\overline{\psi}^{(-1)} = \overline{\Phi}\,\overline{\psi}.$$

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where
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where
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  - S_d is the $\overline{k}(t)$-span of the entries of ψ.
- Once we show this for each d, the equality of transcendence degrees follows (by a comparison of Hilbert series).

Wade's theorem redux

π_q is transcendental

- Work in the setting of the Carlitz motive C with $r = 1$; $\Phi = t - \theta$; $\Psi = \Omega(t)$:

$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

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$$\Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$
- $\mathrm{tr.\,deg}_{\overline{k}(t)}\, \overline{k}(t)(\Omega) = 1$
- The Galois group Γ in this case is $\mathbb{G}_m = \mathrm{GL}_{1/\mathbb{F}_q(t)}$:
$$\mathbb{G}_m(\mathbb{F}_q(t)) = \mathbb{F}_q(t)^\times \cong \mathrm{Aut}_\sigma(\overline{k}(t)(\Omega)/\overline{k}(t))$$
via
$$\gamma \in \mathbb{F}_q(t)^\times,\ h(t,\Omega) \in \overline{k}(t)(\Omega) \quad \Rightarrow \quad \gamma * h(t,\Omega) = h(t,\Omega\gamma).$$

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via
$$\gamma \in \mathbb{F}_q(t)^\times,\ h(t,\Omega) \in \overline{k}(t)(\Omega) \quad \Rightarrow \quad \gamma * h(t,\Omega) = h(t,\Omega\gamma).$$
- Previous theorem $\Rightarrow$
$$\mathrm{tr.\,deg}_{\overline{k}}\, \overline{k}(\Psi(\theta)) = \mathrm{tr.\,deg}_{\overline{k}}\, \overline{k}(\pi_q) = \dim \Gamma = 1.$$