Transcendence in Positive Characteristic

Introduction to Function Field Transcendence

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Outline

Things Familiar

Arithmetic objects from characteristic 0

- The multiplicative group and $\exp(z)$
- Elliptic curves and elliptic functions
- Abelian varieties

The multiplicative group

We have the usual exact sequence of abelian groups

$$0 \to 2\pi i\mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \to 0,$$

where

$$\exp(z) = \sum_{i=0}^{\infty} \frac{z^i}{i!} \in \mathbb{Q}[[z]].$$

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For any $n \in \mathbb{Z}$,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^{\times} \\ {\scriptstyle z \mapsto nz} \downarrow & & \downarrow {\scriptstyle x \mapsto x^n} \\ \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^{\times} \end{array}$$

which is simply a restatement of the functional equation

$$\exp(nz) = \exp(z)^n.$$

Roots of unity

Torsion in the multiplicative group

The *n*-th roots of unity are defined by

$$\mu_n := \{\zeta \in \mathbb{C}^\times \mid \zeta^n = 1\} = \{\exp(2\pi i a/n) \mid a \in \mathbb{Z}\}$$

- $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
- **Kronecker-Weber Theorem:** The cyclotomic fields $\mathbb{Q}(\mu_n)$ provide explicit class field theory for $\mathbb{Q}$.
- For $\zeta \in \mu_n$,

$$\log(\zeta) = \frac{2\pi i a}{n}, \quad 0 \le a < n.$$

Elliptic curves over $\mathbb{C}$

Smooth projective algebraic curve of genus 1.

$$E : y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C}$$

$E(\mathbb{C})$ has the structure of an abelian group through the usual chord-tangent construction.

Weierstrass uniformization

There exist $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, so that if we consider the lattice

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2,$$

then the *Weierstrass $\wp$-function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

The function $\wp(z)$ has double poles at each point in Λ and no other poles.

We obtain an exact sequence of abelian groups,

$$0 \to \Lambda \to \mathbb{C} \stackrel{\exp_E}{\to} E(\mathbb{C}) \to 0,$$

where

$$\exp_E(z) = (\wp(z), \wp'(z)).$$

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Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\exp_E} & E(\mathbb{C}) \\ {\scriptstyle z\mapsto nz}\downarrow & & \downarrow{\scriptstyle P\mapsto [n]P} \\ \mathbb{C} & \xrightarrow{\exp_E} & E(\mathbb{C}) \end{array}$$

where $[n]P$ is the n-th multiple of a point P on the elliptic curve E.

Periods of E

How do we find ω_1 and ω_2?

An elliptic curve E,

$$E : y^2 = 4x^3 + ax + b, \quad a, b \in \mathbb{C},$$

has the geometric structure of a torus in $\mathbb{P}^2(\mathbb{C})$. Let

$$\gamma_1, \gamma_2 \in H_1(E, \mathbb{Z})$$

be generators of the homology of E.

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Then we can choose

$$\omega_1 = \int_{\gamma_1} \frac{dx}{\sqrt{4x^3 + ax + b}}, \qquad \omega_2 = \int_{\gamma_2} \frac{dx}{\sqrt{4x^3 + ax + b}}.$$

Quasi-periods of E

- The differential dx/y on E generates the space of holomorphic 1-forms on E (differentials of the first kind).
- The differential $x\,dx/y$ generates the space of differentials of the second kind (differentials with poles but residues of 0).

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- The differential dx/y on E generates the space of holomorphic 1-forms on E (differentials of the first kind).
- The differential $x\,dx/y$ generates the space of differentials of the second kind (differentials with poles but residues of 0).
- We set

$$\eta_1 = \int_{\gamma_1} \frac{x\,dx}{\sqrt{4x^3 + ax + b}}, \qquad \eta_2 = \int_{\gamma_2} \frac{x\,dx}{\sqrt{4x^3 + ax + b}},$$

  and η_1, η_2 are called the *quasi-periods of E*.
- η_1, η_2 arise simultaneously as special values of the Weierstrass ζ-function and as periods of extensions of E by $\mathbb{G}_a$.

Period matrix of E

- The period matrix of E is the matrix

$$P = \begin{bmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{bmatrix}.$$

It provides a natural isomorphism

$$H^1_{\mathrm{sing}}(E, \mathbb{C}) \cong H^1_{\mathrm{DR}}(E, \mathbb{C}).$$

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- **Legendre Relation:** From properties of elliptic functions, the determinant of P is

$$\omega_1\eta_2 - \omega_2\eta_1 = \pm 2\pi i.$$

Abelian varieties

Higher dimensional analogues of elliptic curves

- An abelian variety A over $\mathbb{C}$ is a smooth projective variety that is also a group variety.
- Elliptic curves are abelian varieties of dimension 1.

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Higher dimensional analogues of elliptic curves

- An abelian variety A over $\mathbb{C}$ is a smooth projective variety that is also a group variety.
- Elliptic curves are abelian varieties of dimension 1.
- Much like for $\mathbb{G}_m$ and elliptic curves, an abelian variety of dimension d has a uniformization,

$$\mathbb{C}^d \big/ \Lambda \cong A(\mathbb{C}),$$

where Λ is a discrete lattice of rank $2d$.

The period matrix of an abelian variety

Let A be an abelian variety over $\mathbb{C}$ of dimension d.

- As in the case of elliptic curves, there is a natural isomorphism,

$$H^1_{\mathrm{sing}}(A, \mathbb{C}) \cong H^1_{\mathrm{DR}}(A, \mathbb{C}),$$

  defined by period integrals, whose defining matrix P is called the *period matrix of A*.

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- We have

$$P = \left[\omega_{ij} \,\middle|\, \eta_{ij}\right] \in \mathrm{Mat}_{2d}(\mathbb{C}),$$

  where $1 \leq i \leq 2d$, $1 \leq j \leq d$.
- The ω_{ij}'s provide coordinates for the period lattice Λ.
- The η_{ij}'s provide periods of extensions of A by $\mathbb{G}_a$.

Things Less Familiar

Transcendence in characteristic 0

- Theorems of Hermite-Lindemann and Gelfond-Schneider
- Schneider's theorems on elliptic functions
- Linear independence results
- Grothendieck's conjecture

Transcendence from $\mathbb{G}_m$

Theorem (Hermite-Lindemann 1870's, 1880's)

Let $\alpha \in \overline{\mathbb{Q}}$, $\alpha \neq 0$. *Then* $\exp(\alpha)$ *is transcendental over* $\mathbb{Q}$.

Examples

Each of the following is transcendental:

- $e \qquad (\alpha = 1)$
- $\pi \qquad (\alpha = 2\pi i)$
- $\log 2 \qquad (\alpha = \log 2)$

Hilbert's Seventh Problem

Theorem (Gelfond-Schneider 1930's)

Let $\alpha, \beta \in \overline{\mathbb{Q}}$, with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$. Then α^β is transcendental.

Examples

Each of the following is transcendental:

- $2^{\sqrt{2}}$ $\quad (\alpha = 2, \beta = \sqrt{2})$
- e^π $\quad (e^\pi = (-1)^{-i})$
- $\dfrac{\log 2}{\log 3}$ $\quad \left(3^{\frac{\log 2}{\log 3}} = 2\right)$

Periods and quasi-periods of elliptic curves

Theorem (Schneider 1930's)

Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$,

$$E : y^2 = x^3 + ax + b, \quad a, b \in \overline{\mathbb{Q}}.$$

- *The periods and quasi-periods of E,*

  $$\omega_1, \omega_2, \eta_1, \eta_2$$

  are transcendental.
- *Let $\tau = \omega_1/\omega_2$. Then either $\mathbb{Q}(\tau)/\mathbb{Q}$ is an imaginary quadratic extension (CM) or a purely transcendental extension (non-CM).*

Linear independence

Linear forms in logarithms

Theorem (Baker 1960's)

Let $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$. *If* $\log(\alpha_1), \ldots, \log(\alpha_m)$ *are linearly independent over* $\mathbb{Q}$, *then*

$$1, \log(\alpha_1), \ldots, \log(\alpha_m)$$

are linearly independent over $\overline{\mathbb{Q}}$.

- Extension of the Gelfond-Schneider theorem ($m = 2$).
- Work of Bertrand, Masser, Waldschmidt, Wüstholz (1970's, 1980's) extended this result to elliptic and abelian integrals.

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Conjecture (Gelfond/Folklore)

Let $\alpha_1, \ldots, \alpha_m \in \overline{\mathbb{Q}}$*. If* $\log(\alpha_1), \ldots, \log(\alpha_m)$ *are linearly independent over* $\mathbb{Q}$*, then they are algebraically independent over* $\overline{\mathbb{Q}}$*.*

Grothendieck's conjecture

Conjecture (Grothendieck)

Suppose A is an abelian variety of dimension d defined over $\overline{\mathbb{Q}}$*. Then*

$$\mathrm{tr.\,deg}(\overline{\mathbb{Q}}(P)/\overline{\mathbb{Q}}) = \dim \mathrm{MT}(A),$$

where $\mathrm{MT}(A) \subseteq \mathrm{GL}_{2d}/\mathbb{Q}$ *is the Mumford-Tate group of A.*

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Let A be an elliptic curve.

- One can show

$$\dim \mathrm{MT}(A) = \begin{cases} 4 & \text{if } \mathrm{End}(A) = \mathbb{Z}, \\ 2 & \text{if } \mathrm{End}(A) \neq \mathbb{Z}. \end{cases}$$

- (G. Chudnovsky, 1970's) If $\mathrm{End}(A) \neq \mathbb{Z}$, then Grothendieck's conjecture is true.

Things Less Less Familiar

- Function fields
- Drinfeld modules
  - The Carlitz module
  - Drinfeld modules of rank 2
- t-modules (higher dimensional Drinfeld modules)
- Transcendence results

Function fields

Let p be a fixed prime; q a fixed power of p.

$$
\begin{array}{ccc}
A := \mathbb{F}_q[\theta] & \longleftrightarrow & \mathbb{Z} \\
k := \mathbb{F}_q(\theta) & \longleftrightarrow & \mathbb{Q} \\
\overline{k} & \longleftrightarrow & \overline{\mathbb{Q}} \\
k_\infty := \mathbb{F}_q((1/\theta)) & \longleftrightarrow & \mathbb{R} \\
\mathbb{C}_\infty := \widehat{\overline{k_\infty}} & \longleftrightarrow & \mathbb{C} \\
|f|_\infty = q^{\deg f} & \longleftrightarrow & |\cdot|
\end{array}
$$

Twisted polynomials

- Let $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be the q-th power Frobenius map: $F(x) = x^q$.
- For a subfield $\mathbb{F}_q \subseteq K \subseteq \mathbb{C}_\infty$, the ring of *twisted polynomials* over K is

$$K[F] = \text{polynomials in } F \text{ with coefficients in } K,$$

subject to the conditions

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subject to the conditions
$$Fc = c^q F, \quad \forall\, c \in K.$$
- In this way,
$$K[F] \cong \{\mathbb{F}_q\text{-linear endomorphisms of } K^+\}.$$
For $x \in K$ and $\phi = a_0 + a_1 F + \cdots a_r F^r \in K[F]$, we write
$$\phi(x) := a_0 x + a_1 x^q + \cdots + a_r x^{q^r}.$$

Drinfeld modules

Function field analogues of $\mathbb{G}_m$ and elliptic curves

Let $\mathbb{F}_q[t]$ be a polynomial ring in t over $\mathbb{F}_q$.

Definition

A *Drinfeld module* over is an $\mathbb{F}_q$-algebra homomorphism,

$$\rho : \mathbb{F}_q[t] \to \mathbb{C}_\infty[F],$$

such that

$$\rho(t) = \theta + a_1 F + \cdots a_r F^r.$$

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- ρ makes $\mathbb{C}_\infty$ into a $\mathbb{F}_q[t]$-module in the following way:

$$f * x := \rho(f)(x), \quad \forall f \in \mathbb{F}_q[t], x \in \mathbb{C}_\infty.$$

- If $a_1, \ldots, a_r \in K \subseteq \mathbb{C}_\infty$, we say ρ is *defined over K*.
- r is called the *rank* of ρ.

The Carlitz module

The analogue of $\mathbb{G}_m$

We define a Drinfeld module $C : \mathbb{F}_q[t] \to \mathbb{C}_\infty[F]$ by

$$C(t) := \theta + F.$$

Thus, for any $x \in \mathbb{C}_\infty$,

$$C(t)(x) = \theta x + x^q.$$

Carlitz exponential

We set

$$\exp_C(z) = z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i} - \theta)(\theta^{q^i} - \theta^q)\cdots(\theta^{q^i} - \theta^{q^{i-1}})}.$$

- $exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is entire, surjective, and $\mathbb{F}_q$-linear.
- Functional equation:

$$\exp_C(\theta z) = \theta \exp_C(z) + \exp_C(z)^q,$$
$$\exp_C(f(\theta)z) = C(f)(\exp_C(z)), \quad \forall f(t) \in \mathbb{F}_q[t].$$

Carlitz uniformization and the Carlitz period

We have a commutative diagram of $\mathbb{F}_q[t]$-modules,

$$\begin{array}{ccc}
\mathbb{C}_\infty & \xrightarrow{\exp_C} & \mathbb{C}_\infty \\
{\scriptstyle z \mapsto \theta z}\downarrow & & \downarrow{\scriptstyle x \mapsto \theta x + x^q} \\
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\end{array}$$

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The kernel of $\exp_C(z)$ is

$$\ker(\exp_C(z)) = \mathbb{F}_q[\theta]\pi_q,$$

where

$$\pi_q = \theta \sqrt[q-1]{-\theta} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}.$$

Wade's result

Thus we have an exact sequence of $\mathbb{F}_q[t]$-modules,

$$0 \to \mathbb{F}_q[\theta]\pi_q \to \mathbb{C}_\infty \xrightarrow{\exp_C} \mathbb{C}_\infty \to 0.$$

Theorem (Wade 1941)

The Carlitz period π_q is transcendental over $\overline{k}$.

Drinfeld modules of rank 2

- Suppose $\rho : \mathbb{F}_q[t] \to \overline{k}[F]$ is a rank 2 Drinfeld module defined over $\overline{k}$ by
$$\rho(t) = \theta + \kappa F + \lambda F^2.$$
- Then there is an unique, entire, $\mathbb{F}_q$-linear function
$$\exp_\rho : \mathbb{C}_\infty \to \mathbb{C}_\infty,$$
so that
$$\exp_\rho(f(\theta)z) = \rho(f)(\exp_\rho(z)), \quad \forall f \in \mathbb{F}_q[t].$$

Periods of Drinfeld modules of rank 2

- Furthermore, there are $\omega_1, \omega_2 \in \mathbb{C}_\infty$ so that

$$\ker(\exp_\rho(z)) = \mathbb{F}_q[\theta]\omega_1 + \mathbb{F}_q[\theta]\omega_2 =: \Lambda,$$

where Λ is a discrete $\mathbb{F}_q[\theta]$-submodule of $\mathbb{C}$ of rank 2.

Periods of Drinfeld modules of rank 2

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  where Λ is a discrete $\mathbb{F}_q[\theta]$-submodule of $\mathbb{C}$ of rank 2.
- **Chicken vs. Egg:**

$$\exp_\rho(z) = z \prod_{0 \neq \omega \in \Lambda} \left(1 - \frac{z}{\omega}\right).$$

- Again we have a uniformizing exact sequence of $\mathbb{F}_q[t]$-modules

$$0 \to \Lambda \to \mathbb{C}_\infty \stackrel{\exp_\rho}{\to} \mathbb{C}_\infty \to 0.$$

Transcendence results for Drinfeld modules of rank 2

Quasi-periods: It is possible to define quasi-periods $\eta_1, \eta_2 \in \mathbb{C}_\infty$ for ρ with the following properties (see notes):

- η_1, η_2 arise as periods of extensions of ρ by $\mathbb{G}_a$.
- Legendre relation: $\omega_1\eta_2 - \omega_2\eta_1 = \zeta\pi_q$ for some $\zeta \in \mathbb{F}_q^\times$.

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Theorem (Yu 1980's)

For a Drinfeld module ρ of rank 2 defined over $\overline{k}$, the four quantities

$$\omega_1, \omega_2, \eta_1, \eta_2$$

are transcendental over $\overline{k}$.

t-modules

Higher dimensional Drinfeld modules

- A *t-module* A of dimension d is an $\mathbb{F}_q$-linear homomorphism,

$$A : \mathbb{F}_q[t] \to \mathrm{End}_{\mathbb{F}_q}(\mathbb{C}_\infty^d) \cong \mathrm{Mat}_d(\mathbb{C}_\infty[F]),$$

  such that

$$A(t) = \theta \mathrm{Id} + N + a_0 F + \cdots a_r F^r,$$

  where $N \in \mathrm{Mat}_d(\mathbb{C}_\infty)$ is nilpotent.

- Thus $\mathbb{C}_\infty^d$ is given the structure of an $\mathbb{F}_q[t]$-module via

$$f * x := A(f)(x), \quad \forall f \in \mathbb{F}_q[t],\ x \in \mathbb{C}_\infty^d.$$

Exponential functions of t-modules

- There is a unique entire $\exp_A : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$ so that

$$\exp_A((\theta \mathrm{Id} + N)z) = A(t)(\exp_A(z)).$$

- If $\exp_A$ is surjective, we have an exact sequence

$$0 \to \Lambda \to \mathbb{C}_\infty^d \stackrel{\exp_A}{\to} \mathbb{C}_\infty^d \to 0,$$

  where Λ is a discrete $\mathbb{F}_q[t]$-submodule of $\mathbb{C}_\infty^d$.
- Λ is called the *period lattice* of A.
- Quasi-periods can also be defined (see notes).

Yu's Theorem of the Sub-t-module

Analogue of Wüstholz's Subgroup Theorem

Theorem (Yu 1997)

Let A be a t-module of dimension d defined over $\overline{k}$. Suppose $u \in \mathbb{C}_\infty^d$ satisfies $\exp_A(u) \in \overline{k}^d$. Then the smallest vector space $H \subseteq \mathbb{C}_\infty^d$ defined over $\overline{k}$ which is invariant under $\theta\mathrm{Id} + N$ and which contains u has the property that

$$\exp_A(H) \subseteq A(\mathbb{C}_\infty),$$

is a sub-t-module of A.

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is a sub-t-module of A.

Theorem (Yu 1997 (Linear independence of Carlitz logarithms))

Suppose $\alpha_1, \ldots, \alpha_m \in \overline{k}$. If $\log_C(\alpha_1), \ldots, \log_C(\alpha_m) \in \mathbb{C}_\infty$ are linearly independent over $k = \mathbb{F}_q(\theta)$, then

$$1, \log_C(\alpha_1), \ldots, \log_C(\alpha_m)$$

are linearly independent over $\overline{k}$.