Brownawell, Chang, and Papanikolas Project

1. Algebraic Independence of Carlitz Periods.

Fix a prime p, and let m be a positive integer. We set

$$\pi_m := \theta \sqrt[p^m-1]{-\theta} \prod_{i=1}^{\infty} \Big(1 - \theta^{1-p^{mi}}\Big)^{-1} \in k_\infty(\sqrt[p^m-1]{-\theta}).$$

In this way $\pi_m = \pi_{p^m}$ in the notes, and π_m is the fundamental period of the Carlitz module for $\mathbb{F}_{p^m}[\theta]$. A question that arises is whether or not these quantities (all of which are in $\mathbb{C}_\infty$) are algebraically independent over $\overline{k} = \overline{\mathbb{F}_p(\theta)}$. The truth is that this problem has been solved by Laurent Denis:

Theorem 1 ([Denis 1998])**.** *The numbers*

$$\pi_1, \pi_2, \pi_3, \ldots$$

are algebraically independent over $\overline{k}$*.*

Denis' proof utilizes a method of Mahler in a very clever way. What this project entails is exploring an alternate method for proving Denis' theorem. More specifically, we will try to prove this theorem using methods in [Papanikolas 2008] (see also [Pellarin 2007]). As it will be easier, we will consider the problem of showing that the three numbers

$$\pi_1, \pi_2, \pi_3$$

are algebraically independent. Hopefully this will provide sufficient information for the general proof. Intrepid readers can also consult [Chang-Yu].

2. Defining Equations for Galois Groups Associated to Carlitz Zeta Values.

Fix $q = p^m$, and let $\pi = \pi_q$ be the period of the Carlitz module for $\mathbb{F}_q[\theta]$. For an integer $s \geq 1$, we define the Carlitz zeta value

$$\zeta_C(s) = \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ a \text{ monic}}} \frac{1}{a^s}.$$

We note first that the sum for $\zeta_C(s)$ converges to an element of k_∞, since the degrees of the denominators go to ∞. In fact, because k_∞ is non-archimedean, $\zeta_C(1)$ makes sense.

These zeta values, first studied by Carlitz [Carlitz 1935], share many interesting properties with special values of the Riemann zeta function. See [Goss], [Thakur], [Rosen] for more general information. See [Anderson-Thakur 1990], [Chang-Yu], and [Yu 1991] for more specific information related to this project.

Euler-Carlitz Relations: Suppose that $(q-1) \mid s$. Then there is $R_s \in \mathbb{F}_q(\theta)$ so that

$$\zeta_C(s) = R_s \pi^s.$$

In this sense we say that a zeta value $\zeta_C(s)$ is "even" if s is a multiple of $q-1$ (and "odd" otherwise). The proof of this formula is in the references listed above, and for specific information about the constant R_s, see [Anderson-Thakur 1990] or [Thakur, Thm. 5.2.1].

The project will be the following: (The following description will make a little bit more sense once the rest of the notes are available.) Since $\zeta_C(s)$ and π^s arise as periods of a t-motive X that lies in an extension

$$0 \to C^{\otimes s} \to X \to \mathbf{1}^h \to 0,$$

the Euler-Carlitz relations should induce a relations on the Galois group Γ of X. What are these relations? We'll look at $\zeta_C(q-1)$ first and perhaps $\zeta_C(2(q-1))$. Things get fairly complicated after that.