Arithmetic of values of E-functions and G-functions

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The subject

Suppose we have an analytic function $f(z)$ with power series expansion $\sum_{n\ge0} f_n z^n$ whose coefficients are contained in an algebraic number field, or even $\mathbb{Q}$. It is a natural question to ask about the irrationality or transcendence of values of f algebraic or rational points. The standard example is the exponential function e^z whose values at non-zero algebraic points are known to be transcendental.
In order to be able to say anything at all, a number of assumptions have to be made about f. Siegel's theory of transcendence (started in 1929) provides us with two such classes, the E-functions and G-functions. Both of them satisfy a linear differential equation polynomial coefficients. The E-functions have a power series expansion which is reminiscent of the exponential function. For the case of rational coefficients f_n they look like

$$\sum_{n\ge0} \frac{a_n}{n!} z^n$$

with $a_n \in \mathbb{Q}$ and there exists $C > 0$ such that both $|a_n|$ and the common denominator of $a_0, \ldots, a_n$ are bounded by C^n for all $n \ge 0$. The transcendence theory of the values of E-functions is quite complete. Siegel also introduced G-functions, having a power series $\sum_{n\ge0} a_n z^n$ where the a_n again satisfy the same assumptions as above. They are reminiscent of the geometric series $\sum_n z^n$, hence the name G-functions. The best known example of a G-function is the so-called Gauss hypergeometric function

$${}_2F_1(a, b, c|z) = \sum_{n\ge0} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ and $a, b, c \in \mathbb{Q}$. This power series has radius of convergence of 1. Clearly, E-functions and G-functions are closely related, but the transcendence theory for G-functions is very rudimentary. In fact, G-functions may have algebraic values at non-zero algebraic points. Since algebraic functions (over $\mathbb{Q}(z)$) are also G-functions, this is no surprise, but there exist interesting evaluations of transcendental G-functions as well. A striking example is

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}, \frac{1}{2}\middle| \frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11}.$$

During the lectures we shall briefly discuss the transcendence and irrationality results for values of E-functions and G-functions. As a second goal we like to discuss the question how they can be classified. For example, the arithmetic conditions on these functions influence the analytic properties of the differential equation which they satisfy. For G-functions there are some fundamental results in this direction by N.Katz and the Choodnovsky's, which will be discussed during the lectures. For E-functions we explain Y.André's beautiful work which provides us with a new transcendence technique. Although Siegel first wondered about it, it is still not known what a G-function or E-function really is. Time permitting we may be able to discuss some ideas about this.

The projects

Explicit one variable G-functions

Let $k_1, \ldots, k_r$ and $l_1, \ldots, l_s$ be positive integers such that $k_1 + \cdots + k_r = l_1 + \cdots + l_s$. Then the series $\sum_{n\geq 0} a_n z^n$ with

$$a_n = \frac{(k_1 n)! \cdots (k_r n)!}{(l_1 n)! \cdots (l_s n)!}$$

is a G-function.

Problem Determine those combinations of k_i, l_j such that a_n is an integer for all n.

This is, to some extent, an open problem and it is not clear what the answer is. There are a number of very nice partial results and we shall describe them here. A first observation is the following exercise, which presents itself immediately. It is also known as Landau's criterion

Exercise 1. Denote $\theta(\mathbf{k}, \mathbf{l}, x) := \sum_{i=1}^{s}\{l_i x\} - \sum_{j=1}^{r}\{k_j x\}$ where $\mathbf{k} = (k_1, \ldots, k_r), \mathbf{l} = (l_1, \ldots, l_s)$ and $\{x\}$ denotes x minus the largest integer not exceeding x. Show that the numbers a_n defined above are all integers if and only if $\theta(\mathbf{k}, \mathbf{l}, x) \geq 0$ for all $x \in \mathbb{R}$.

Some other easy properties are the following,

Exercise 2 Show,

- $\theta(\mathbf{k}, \mathbf{l}, x)$ is locally constant.
- $\int_0^1 \theta(\mathbf{k}, \mathbf{l}, x)dx = (s - r)/2$.
- $a_n \in \mathbb{Z}$ for all n implies that $s \geq r$.
- $a_n \in \mathbb{Z}$ for all n and $s = r$ is only trivially possible, i.e. $\mathbf{k} = \mathbf{l}$.

Some of these things can be found in F.Rodriguez-Villegas draft "Hypergeometric families of CY threefolds". A beautiful observation by F.Rodriguez-Villegas (see the link "Integral ratios..." on the webpage),

Theorem The generating function $\sum_{n\geq 0} a_n z^n$ is algebraic if and only if $a_n \in \mathbb{Z}$ for all n and $s - r = 1$.

This work is based on work by Beukers-Heckman, Invent.Math 95 (1989), 325-354, which you find summarised on these webpages. The full list of $\mathbf{k}, \mathbf{l}$ satisfying Rodriguez-Villegas criterion could be provided using the list given in the Inventiones paper. However, that list is constructed by using properties of complex reflection groups. It would be a challenge to recover a full list based on other (number theoretical) principles.
Of course it is an even bigger challenge to find a full answer to the above Problem. Of course, when you have no other ideas one could try to start with the case $r = 2$ (the case $r = 1$ being trivial).

Explicit two variable G-functions

An even bigger challenge is the case when we have linear forms $k_1(m,n), \ldots, k_r(m,n)$ and $l_1(m,n), \ldots, l_s(m,n)$ in 2 variables m, n with non-negative integer coefficients and such that $\sum_i k_i(m,n) = \sum_j l_j(m,n)$. Define

$$b(m,n) = \frac{k_1(m,n)! \cdots k_r(m,n)!}{l_1(m,n)! \cdots l_s(m,n)!}.$$

One might think of these numbers as coefficients of a 2-variable G-function. The question is to find non-trivial examples of forms k_i, l_j such that $b(m,n)$ is an integer for all $m, n \geq 0$, and see how far one can get.
Of course Exercises 1 and 2 from the previous subsection could be carried out here too. In this case no complete lists are known (with a number of exceptions that we may discuss).
Here are some non-trivial examples which can be found in P.A. Picon, European J. of Combinatorics 1994 (15), 561-577, which can also be downloaded from the p;resent pages.

$$\frac{(5m)!(5n)!}{(2m)!(2n)!(2m+n)!(2n+m)!}$$

on page 571 and another (admittedly 3D)

$$\frac{(3m+3n)!(5m+2k)!(5n+2k)!}{(2m+2n+k)!^2(m+n+k)!m!^3n!^3k!}.$$

Algebraic values of hypergeometric functions

This requires some background in the theory of hypergeometric functions and their monodromy properties. Based on classical identities and evaluations of hypergeometric functions one can produce a examples of evaluations at rational arguments which are algebraic. Some examples,

$${}_2F_1(1-3a, 3a, a|1/2) = 2^{2-3a}\cos\pi a, \qquad a \in \mathbb{Q}$$

$${}_2F_1\left(\frac{7}{48}, \frac{31}{48}, \frac{29}{24}\middle| -\frac{1}{3}\right) = 2^{5/24} \cdot 3^{-11/12} \cdot 5\sqrt{\frac{\sin\pi/24}{\sin 5\pi/24}}.$$

The purpose of this project is to find an overview, as complete as possible, of such evaluations.