Bernd Sturmfels'
Arizona Lecture #2
Discriminants & Resultants

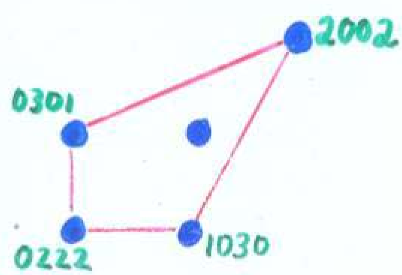
For a cubic polynomial

$$f(t) = x_1 + x_2 t + x_3 t^2 + x_4 t$$

the discriminant equals

$$\Delta = 27x_1^2x_4^2 - 18x_1x_2x_3x_4 + 4x_1x_3^3 + 4x_2^3x_4 - x_2^2x_3^2.$$

The Newton polytope of Δ is a quadrangle



The A-Discriminant

$A \in \mathbb{Z}^{d\times n}$ $\quad$ $\mathrm{rank}(A) = d$

$(1,1,\dots,1) \in \mathrm{rowspace}(A)$

The matrix A represents a family of hypersurfaces in $(\mathbb{C}^*)$ defined by the Laurent polynomial

$$f(t) = \sum_{j=1}^{n} x_j \cdot t_1^{a_{1j}} t_2^{a_{2j}} \cdots t_d^{a_{dj}}.$$

Consider the set of all points $(x_1 : x_2 : \cdots : x_n) \in \mathbb{P}_{\mathbb{C}}^{n-1}$ such that the hypersurface $\{f(t) = 0\}$ has a singular point in $(\mathbb{C}^*)^d$...

The closure of this set is an irreducible variety in $\mathbb{P}_{\mathbb{C}}^{n-1}$, denoted Δ_A and called the A-discriminant.

Often - **but not always** - Δ_A is a hypersurface, defined by an irreducible polynomial $/\mathbb{Z}$.

Example 1
$d=2,\ n=3$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$f = x_1 t_2^2 + x_2 t_1 t_2 + x_3 t_1^2$$

$$\Delta_A = x_2^2 - 4x_1x_3$$

Discriminant of a binary form

Computing the A-discriminant:

.... for instance, in Macaulay 2:

- Consider all partial derivatives of $f(t_1, \dots, t_d)$

$$\left\langle \frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, \dots, \partial f / \partial t_d \right\rangle$$

- This is an ideal in $d+n$ variables $t_1, \dots, t_d > x_1, \dots, x_n$.

- Eliminate $t_1, \dots, t_d$ to get

$$\Delta_A(x_1, x_2, \dots, x_n)$$

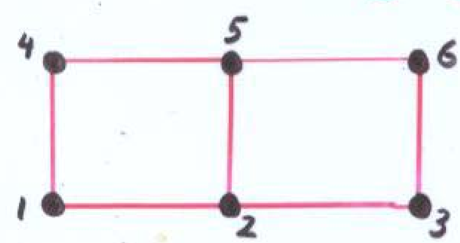
TRY IT TONITE

The Discriminant of a Rectangle....

$d=3$

$n=6$

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



$$f = x_1 t_2 + x_2 t_1 t_2 + x_3 t_1^2 t_2$$
$$+ x_4 t_3 + x_5 t_1 t_3 + x_6 t_1^2 t_3$$

To compute Δ_A, we take derivatives

$$\partial f / \partial t_1 = x_2 t_2 + 2 x_3 t_1 t_2 + x_5 t_3 + 2 x_6 t_1 t_3$$

$$\partial f / \partial t_2 = x_1 + x_2 t_1 + x_3 t_1^2$$

$$\partial f / \partial t_3 = x_4 + x_5 t_1 + x_6 t_1^2$$

... is the Sylvester Resultant

$$\Delta_A = Res_{t_1}(x_1 + x_2 t_1 + x_3 t_1^2, x_4 + x_5 t_1 + x_6 t_1^2)$$

$$= \det \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ 0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & 0 \\ 0 & x_4 & x_5 & x_6 \end{bmatrix}$$

This is a polynomial of degree 4 in 6 unknowns having 7 terms.

Can You draw its tropical hypersurface?

Punchline:

.... discriminants → resultants → discrimina

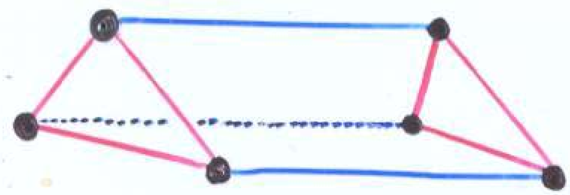
.. chickens → eggs → chickens → ....

Determinantal Varieties

$d=4$
$n=6$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Product of two simplices



a.k.a TOBLERONE

... family of bilinear forms

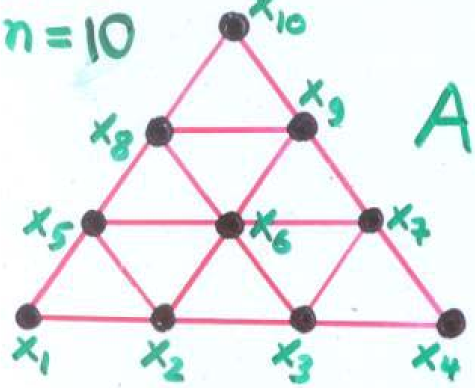
$$f = (t_1\ t_2)\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}\begin{pmatrix} t_3 \\ t_4 \\ t_5 \end{pmatrix}$$

The A-discriminant Δ_A is the codimension two variety of all rank one matrices $\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix}$.

Elliptic Curves

$d = 3$

$n = 10$



$$A = \begin{bmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$f =$ homogeneous cubic polynomial in t_1, t_2, t_3

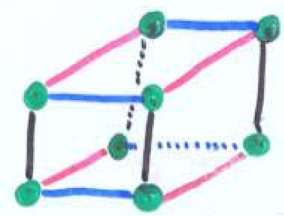
The discriminant Δ_A is a polynomial of degree 12 in 10 unknowns $x_1, x_2, \ldots, x_{10}$ which vanishes if and only if the plane cubic $\{f = 0\}$ has a singular point.

The discriminant Δ_A has 2040 monomials

$2\times2\times2$ - Hyperdeterminant

$$f = \sum_{i,j,k=0}^{1} x_{ijk}\, t_i^{(1)} t_j^{(2)} t_k^{(3)}$$

A = the 3-cube



$$\Delta_A = 4x_{000}x_{011}x_{101}x_{110} + 4x_{001}x_{100}x_{010}x_{111}$$
$$+ x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{100}^2 x_{011}^2$$
$$- 2x_{000}x_{111}x_{001}x_{110} - 2x_{000}x_{111}x_{010}x_{101}$$
$$- 2x_{000}x_{111}x_{100}x_{011} - 2x_{001}x_{110}x_{010}x_{101}$$
$$- 2x_{001}x_{110}x_{100}x_{011} - 2x_{010}x_{101}x_{100}x_{011}$$

Physicists call this the tangle ...

$2\times2\times2\times2$-Hyperdeterminant

$$f = \sum_{i,j,k,l=0}^{1} x_{ijkl}\, t_i^{(1)} t_j^{(2)} t_k^{(3)} t_l^{(4)}$$

A = the 4-cube

The A-discriminant Δ_A is the hyperdeterminant of the tensor (x_{ijkl}). It has degree 24 and is the sum of 2,894,276 monomials.

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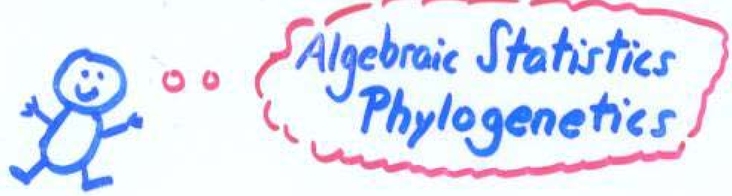

Can You "draw" the Newton polytope of Δ_A?
It has only 25,448 vertices ....

What is wrong with all of these examples?

A: They are misleading because they are **too easy**.

Q: Are you ↯? What do you mean? 

A: In each case, the underlying **toric variety** X_A is **smooth**.

In APPLICATIONS OF ALGEBRAIC GEOMETRY we encounter **arbitrary matrices** A.

Q: Wasn't this all solved by **Gel'fand - Kapranov - Zelevinsky**?

The famous green book [GKZ 1994]

- All classical resultants and discriminants are A-discriminants
- The Newton polytope of Δ_A is a Minkowski summand of the secondary polytope of Δ_A
- An alternating degree formula for Δ_A in the special case when the toric variety X_A is smooth.
- Techniques are quite advanced and give little information when X_A is not smooth or $\mathrm{codim}(\Delta_A) > 1$

Mixed Discriminants

.... characterize systems of s equations in s unknowns that have a double root.

Here $d = 2s$

n = total number of terms

RUNNING EXAMPLE $d=4, \; n=8$

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 \\ 53 & 47 & 43 & 41 & 37 & 31 & 29 & 23 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

What does Δ_A mean?

And how to compute it?