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DISCRIMINANTS, RESULTANTS AND THEIR TROPICALIZATION

BERND STURMFELS

The aim of this course is to introduce discriminants and resultants, in the sense of Gel'fand, Kapranov and Zelevinsky [3], with emphasis on the tropical approach which was developed by Dickenstein, Feichtner and the lecturer [2].

Lecture 1: A-discriminants. Every configuration A of lattice points defines a projective toric variety X_A, whose dual variety X_A^* is typically a hypersurface, known as the A-*discriminant*. We show how many classical discriminants and classical resultants arise as special cases of this construction.

Lecture 2: Degree Formulas. We present various known formulas for the degree and Newton polytope of the A-discriminant. In the case of resultants, this degree involves mixed volumes [5], and is closely related to determinantal formulas for eliminating variables from systems of polynomial equations. For arbitrary A-discriminants, a positive degree formula was recently given in [2].

Lecture 3: Tropical Varieties. This lecture assumes familiarity with matroids and Gröbner bases, and it gives an otherwise self-contained introduction to tropical algebraic geometry. Software tools for computing arbitrary tropical varieties will be discussed briefly [1]. We then show how to compute the degree and the toric degenerations of a projective variety from its tropicalization, and how to tropicalize the image of a map given by monomials in linear forms.

Lecture 4: Tropical Horn Uniformization. Kapranov's Horn uniformization [4] parametrizes the A-discriminant by monomials in linear forms. From this we derive that the tropical A-discriminant is the Minkowski sum of the co-Bergman fan of A and the row space of A. This explains the degree formulas discussed in Lecture 2, and it gives an algorithm for computing its Newton polytope. We also relate this to the combinatorial aspects of Gel'fand-Kapranov-Zelevinsky theory (regular triangulations, secondary polytopes [3]).

Project: Mixed Discriminants. Given a sparse system of n polynomials in n variables with indeterminate coefficients, their *mixed discriminant* is the unique irreducible polynomial in the coefficients which vanishes when the system has a double root. The aim of this project is to find a formula for the degree and (Newton polytope) of the mixed discriminant, at least when $n = 2$.

As an example consider the following system of two equations in x and y:

$$\begin{aligned} c_1x^2y^{53} + c_2x^3y^{47} + c_3x^5y^{43} + c_4x^7y^{41} &= 0, \\ c_5x^{11}y^{37} + c_6x^{13}y^{31} + c_7x^{17}y^{29} + c_8x^{19}y^{23} &= 0. \end{aligned}$$

If the coefficients $c_1, c_2, \ldots, c_8$ are random complex numbers then this system has ??? distinct roots in $(\mathbb{C}^*)^2$. The vanishing of the mixed discriminant is the condition for this system to have a double root. It is a homogeneous polynomial of degree ??? in the unknowns $c_1, c_2, \ldots, c_8$. Can **you** figure out what the two integers indicated by the question marks "???" are ? If yes, then this AWS student project is the one for **you**. To get our discussions started, please e-mail me your answers right away to `bernd@math.berkeley.edu`.

References

[1] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas: *Computing tropical varieties*; preprint, `http://front.math.ucdavis.edu/math.AG/0507563`.

[2] A. Dickenstein, E.-M. Feichtner and B. Sturmfels: *Tropical Discriminants*; in preparation, preprint will appear on the `ArXiV` in October 2005.

[3] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky: *Discriminants, Resultants, and Multidimensional Determinants*; Birkhäuser, Boston, MA, 1994.

[4] M.M. Kapranov: *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*; Mathematische Annalen 290 (1991), no. 2, 277–285.

[5] B. Sturmfels: *On the Newton polytope of the resultant*; Journal of Algebraic Combinatorics 3 (1994), 207–236.