Cohomology by approximation

Let M be a f.g. graded S-module, with minimal resolution:

$$0 \to F_r \longrightarrow \cdots \longrightarrow F_1 \to F_0 \to M \to$$

where: $$F_i = \bigoplus_{j=0}^{b_i} S(-a_{ij})$$

Def (Castelnuovo-Mumford regularit

$$\operatorname{reg} M := \max_{i,j} \{a_{ij} - i\}$$

Example I = ratl. quartic

$$0 \to S(-5) \longrightarrow S(-4)^4 \longrightarrow \begin{matrix} S(-2) \\ \oplus \\ S(-3) \end{matrix}$$

$\operatorname{reg} I = 3$

$\operatorname{reg} S/I = 2$ $\qquad 5-2 \qquad 4-1 \qquad 3$

theorem (Serre)

Let $i \geq 0$ be an integer.

For all $\ell \geq \operatorname{reg}(M) - i$

$$H^i_*(\widetilde{M})_{\geq 0} \simeq \operatorname{Ext}^i_S(J_\ell, M)_{\geq 0}$$

where $J_\ell = (x_0^\ell, \ldots, x_n^\ell)$

'**proof**'

check: true for M free.

use induction on projective dimension of M, and 5-lemma.

Improving the module M

- if $\operatorname{pdim}(M) \leq n-1$,

  $M = H^0_{\mathfrak{m}}(M)$

  can't improve M

- if $\operatorname{pdim}(M) = n+1$

  $M = F/I$ $\quad F$ free

  $$I^{sat} := I : (x_0, \dots, x_n)^\infty$$
  $$= \{ m \in F : x_i^N m \in I \text{ all } i, N \gg$$

  then $\widetilde{M} = F/I^{sat}$

  $\operatorname{pd} \leq n$ now

- if pdim $M = n$

$$\mathrm{Hom}_S((x_0^{\ell}, \ldots, x_n^{\ell}), M)$$

has same sheaf as M

often better (or best) presentation.

[any ℓ, $\ell \geq \operatorname{reg} M$ is best]

Example

$$X = V(a^5 + b^5 + c^5 + d^5 + e^5) \subseteq \mathbb{P}^4$$

quintic 3-fold

consider $(\Omega^1_X)^{\otimes 2}$

suppose $\Omega^1_X = \widetilde{M}$ omegaX

$\boxed{M \otimes M}$ om2

$$\begin{matrix} S(-5)^{200} \\ \oplus \\ S(-9)^{100} \end{matrix} \longrightarrow S(-4)^{100} \longrightarrow M \otimes M \to 0$$

om2sat

om2depth2 $H^0_*(\widetilde{M \otimes M})$

Line bundles and divisors

line bundle on X

$\equiv$ locally free rank 1 coherent sheaf on X

<u>example</u> Let $X \subseteq \mathbb{P}^n$ smooth

$D \subseteq X$ irreducible codim 1

with ideal $J \subset R = S/I$

$$\mathcal{O}_X(-D) := \widetilde{J}$$

$$\mathcal{O}_X(D) := \widetilde{\mathrm{Hom}_R(J, R)}$$

[D is locally defined by one equation]

Divisors on X

$D = \sum n_i D_i$ formal sum

$n_i \in \mathbb{Z}$

$D_i \subset X$

irred, codim 1

can build $\mathcal{O}_X(-D)$, $\mathcal{O}_X(D)$

Suppose $\mathcal{O}_X(D) = \widetilde{M}$

$\mathcal{O}_X(E) = \widetilde{N}$

M, N $\underline{\underline{R}}$-modules

then

$$\mathcal{O}_X(D+E) = \widetilde{M \otimes_R N}$$

$$\mathcal{O}_X(-D) = \widetilde{\mathrm{Hom}_R(M,R)}$$

$$\mathcal{O}_X(D-E) = \widetilde{\mathrm{Hom}_R(N,M)}$$

Linear equivalence

Def $D \sim E \iff \mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$

$$\iff \mathcal{O}_X(D-E) \simeq \mathcal{O}_X$$

problem Given M an R-module.

decide: is $\widetilde{M} \simeq \mathcal{O}_X$

solution find $H^0_{\geq 0}(\widetilde{M})$

Yes, if this is $H^0_{\geq 0}(\widetilde{R})$

No, otherwise

Divisors on a curve X

$$D = \sum n_i p_i \qquad n_i \in \mathbb{Z}, \quad p_i \in X$$

$$\deg D = \sum n_i$$

Riemann-Roch

If $L \simeq \widetilde{M}$ is a line bundle on X, then

$$\deg L = \underbrace{\chi(\widetilde{M}) - \chi(\mathcal{O}_X)}_{\text{Hilbert functions!}}$$

Divisors on a surface X

Suppose $C, D \subset X$

are irreducible curves

problem: find intersection

number $C \cdot D$

same problem: C, D divisors

same problem: $\tilde{M}, \tilde{N}$ line bundles

Def

$$\tilde{M} \cdot \tilde{N} := \chi(\mathcal{O}_X) - \chi(\tilde{M}) - \chi(\tilde{N}) + \chi(\tilde{M} \otimes \tilde{N})$$

essentially Riemann-Roch for surfaces

Example

$$X \subseteq \mathbb{P}^3$$
$$\parallel$$
$$V(a^3+b^3+c^3+d^3)$$

$L_1 = V(a+b\,,\, c+d)$

$L_2 = V(a+c\,,\, b+d)$

$L_3 = V(a+d\,,\, b+c)$

$L_1^2 = -1$

$L_2^2 = -1$

$L_1 \cdot L_2 = 1$

can recover the line L_1

from $\mathcal{O}_X(L_1) = \widetilde{M}$

since $H^0(\widetilde{M}) = k$.

$\mathcal{O}_X(L_1+L_2+L_3) = \mathcal{O}_X \qquad L_1+L_2+L_3 \sim ($

Canonical bundle

Serre duality:

If $X \subseteq \mathbb{P}^n$ is Cohen-Macaulay (e.g. smooth), then $\exists$ a locally free sheaf ω_X s.t. for all locally free $\mathcal{F}$ on X

$$H^i(\mathcal{F}) \simeq H^{d-i}(\omega_X \otimes \mathcal{F}^*)'$$

where $d = \dim X$

$$\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$$

useful:

$$\boxed{H^0_*(\omega_X) = \mathrm{Ext}^c_S(S/I, S(-n-1))}$$

<u>proof</u>

Serre duality also says:

$$H^i_*(\mathcal{F}) \simeq H^{d-i}_*(\omega_X \otimes \mathcal{F}^*)^\vee$$

$$\therefore \quad H^0_*(\omega_X) \simeq H^d_*(\mathcal{O}_X)^\vee$$

$$\simeq \mathrm{Ext}^c_S(S/I, S(-n-1))$$

$$H^i(\mathcal{F}(\overset{e}{\cancel{d}})) \simeq H^{d-i}(\omega_X \otimes \mathcal{F}^* \otimes \mathcal{O}(\overset{-e}{\cancel{-d}}))^\vee$$

<u>so</u>

$$\boxed{\omega_X = \mathrm{Ext}^c_S(S/I, S(-n-1))}$$

$c = \mathrm{codim}\, X \subseteq \mathbb{P}^n$

$X \subseteq \mathbb{P}^n$ surface

Then X is rational

$\Longleftrightarrow \quad H^0(\omega_X^{\otimes 2}) = H^1(\mathcal{O}_X) = 0$

Mystery surface

$X \subseteq \mathbb{P}^4$ codim 2
degree 6

K_X = canonical bundle

$H^0(K_X) = 0$ $K_X \cdot H = -2$

$K_X^2 = -1$

$h^1(\mathcal{O}_X) = 0$ $h^0(K_X^{\otimes 2}) = 0$

so X is rational

— — — — —

$|H+K|$
$|H+2K| = \phi$ $\Rightarrow$ $C \in |H+K|$ has every comp rational

C deg 4 quartic ratl curve

$C^2 = 1$ $h^0(\mathcal{O}_X(C)) = 3$

Linear systems

$L = \widetilde{M}$ line bundle on X,

$L = \mathcal{O}_X(D)$

$f \in H^0(L)$

$\Longleftrightarrow$

$C \in |D| = \{ D + \mathrm{div}(f) : f \in H^0(L) \}$

problem:

Given $f \in H^0(\widetilde{M})$ (or $f \in M_0$)

find $C \subset X$

solution $M^* := \mathrm{Hom}_R(M, R)$

then $\widetilde{M} = \widetilde{M^{**}}$

but $M^{**} = \mathrm{Hom}_R(M^*, R)$

If $f \in M_0^{**}$ then

$C \subset X$ has ideal $\mathrm{image}(f)$

X is

$\mathbb{P}^2$ blown up at 10 points

embedded by

$$H = 4L - E_1 - \cdots - E_{10}$$

$$K = -3L + E_1 + \cdots + E_{10}$$

$$X \longrightarrow \mathbb{P}^2$$

corresponds to $|C|$

Bordiga surface

What did you leave out, Stillman?

- Bernstein - Gelfand - Gelfand via Eisenbud - Floystad - Schreyer gives great method to find cohomology
- $\mathrm{Ext}^i_X(\widetilde{M}, \widetilde{N}) = \mathrm{Ext}^i_R(M_{\geq d}, N)_0$ for $d \gg 0$ (G. Smith)
- toric varieties (Eisenbud - Mustata - S)
- $f : X \subset \mathbb{P}^n \times \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, A$

  $R^i f_* (\mathcal{F})$ (Eisenbud - Schreyer)