**Computing with cohomology in algebraic geometry**

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In modern algebraic geometry, cohomology is an important tool for many different kinds of problems: it gives rise to numerical invariants of algebraic varieties and it may be used to find tangent spaces of deformation spaces and parameter spaces, among many other applications.

In this series of lectures, we introduce computing sheaf cohomology on projective varieties and schemes, and illustrate its use in examples and computational applications. Our main goals are (1) to bring to life the abstract cohomology machinery by learning how to compute cohomology and use it computationally in examples, and (2) to present the state of the art methods for computing sheaf cohomology.

The suggested project (described later) will have us working on an unsolved problem, that is interesting by itself, but it also ties into the larger goal of "mapping the Hilbert scheme".

**Lecture 1**: We show how to think of sheaves on projective schemes in a computational manner, and show how to compute the most important sheaves associated to a projective variety: tangent and cotangent sheaves, normal sheaves, canonical sheaves, and line bundles and divisors.

It is amazing that one can compute all of these, using only Groebner bases and some byproducts of computing Groebner bases.

**Lecture 2**: The Cech cohomology method of defining sheaf cohomology translates into a reasonable method for computing sheaf cohomology. We describe Serre's methods for computing sheaf cohomology: as direct limits of Ext modules, and via local duality as the dual of an Ext module. We apply these methods to some of the sheaves that we considered in the first lecture.

**Lecture 3**: Each sheaf on projective space gives rise by an explicit method to an exact complex (infinite in both directions) of free E-modules, where E is the exterior algebra. (This is called the Berstein-Gelfand-Gelfand correspondence). The resulting complex is called the Tate resolution, and can be computed using computer algebra systems, such as Macaulay2. It has many amazing properties and applications, including that the cohomology of the original sheaf and all of its twists appears in the Tate resolution! This turns out to be an excellent method to compute sheaf cohomology. This approach was pioneered by Eisenbud, Floystad and Schreyer. In this lecture we present this technique and its application to computing sheaf cohomology.

**Lecture 4**: We apply the above technique to examples. Depending on the interest and background of the students, we will consider (1) the Beilinson monad, which is a very interesting method which can be used to (attempt to) construct varieties with specific cohomology, (2) resultants and Chow varieties, (3) computing higher direct image sheaves, and applications, or (4) computing the "Hodge diamond" of a variety.

**Reading list and prerequisites**

A knowledge of sheaves and schemes is not necessary for these lectures, nor for the project, The only prerequisite as far as this goes is an understanding that cohomology of sheaves is important and carries interesting geometric information.

In order to prepare for these lectures, I suggest reading about Groebner bases and some of their applications (especially syzygies and free resolutions). An excellent readable introduction can be found in the book "Ideals, Varieties, and Algorithms", by Cox, Little and O'Shea.

Since we will be using Macaulay2 throughout the lectures and projects, it is worth downloading the latest version from our web site: `http://www.math.uiuc.edu/Macaulay2` and playing with it. Closer to the winter school, I will place a Macaulay 2 tutorial on the winter school web site, along with some exercises for you to develop familiarity with the system.

Although knowledge of sheaves is not completely necessary, it is useful. I suggest Serre's FAC (Faisceaux Algebrique Coherent) paper from 1955 (this one is in French). Hartshorne's book "Algebraic Geometry" has a good introduction to sheaves, at the beginning of Chapter 2. However, our view of sheaves will be far more explicit and computational. So even if you don't know much about sheaves, these lectures should be understandable. The most important part is an understanding that sheaves and their cohomology is important in the first place!

**Suggested Project**

If $I$ is a homogeneous ideal in the polynomial ring $S = k[x_0, ..., x_n]$, then the normal sheaf $N$ of the projective variety $X = V(I) \subset \mathbf{P}^n$ corresponds to the graded $S$-module $Hom(I/I^2, S/I)$. The normal sheaf contains a large amount of information about "nearby" varieties (i.e. deformations).

In the special, yet very interesting case when $I$ is an ideal generated by monomials, we will attempt to find formulas for the dimensions of the cohomology groups $\dim H^0(N)$ and $\dim H^1(N)$ for certain classes, or certain examples of monomial ideals. No general formulas for these dimensions are known.

Besides being an open problem that can be attacked using computational methods, these dimensions are interesting in "mapping the Hilbert scheme". The Hilbert scheme is a projective algebraic set (scheme) whose points are in 1-1 correspondence with homogeneous (and saturated) ideals $I$. A path on the Hilbert scheme is a deformation, or family of varieties. For example, a Groebner basis computation gives rise to a path which connects your original ideal to its initial ideal of monomials. Because of this, it is very interesting to understand how the monomial ideals sit on the Hilbert scheme. They are in some ways the "backbone" or "subway stops" which allow you to move around the Hilbert scheme. The cohomology groups we will consider in this project give geometric information about how the ideal sits on the Hilbert scheme.

For example, $\dim H^0(N)$ is the dimension of the Zariski tangent space at the point of the Hilbert scheme at the point corresponding to the ideal $I$. $H^1(N)$ contains the obstructions for this Hilbert scheme to be smooth at this point (so if it is zero, the Hilbert scheme is smooth at that point). In general, cohomology groups carry interesting and subtle geometric information, in this case about deformations of the algebraic variety $X$.

**Other possible projects**

There are many other projects that one could suggest, depending on the background and interests of the participants. For example, suppose you are given equations for a "mystery" variety, and you wish to determine some structural information about it (e.g. If it is a surface, where does it fit in the Kodaira classification of surfaces. Or, if it is a rational surface, how is it obtained from $\mathbf{P}^2$ by a series of blow-up and blow-downs of points). In this possible project, we would be given equations for a variety, and we would compute its cohomology, and attempt to understand the structure of the variety.