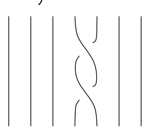
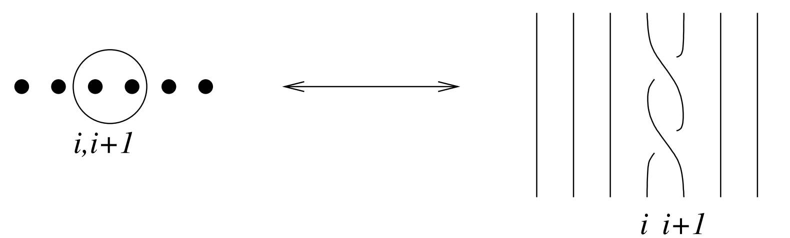
Recall from review session

$$\Gamma_{0,n} = \mathrm{Diff}^+(\Sigma_{0,n})/\mathrm{Diff}^\circ \cong \pi_1(M_{0,n})$$

where

$$\Sigma_{0,n} = \text{top. sphere with n ordered marked points}$$
$$\mathrm{Diff}^+ = \text{oriented diffeomorphisms fixing the points}$$
$$\mathrm{Diff}^\circ = \text{those isotopic to the identity}$$

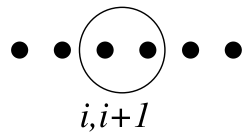
and we saw that $\Gamma_{0,n} = B_n^{\mathrm{pure}}/\text{stuff}$ where B_n^{pure} denotes the group of n-stranded braids  with each strand ending in its starting position.

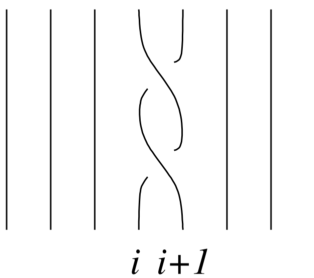
The Dehn twist along the loop 

braid

$$\Gamma_{0,[n]} = \mathrm{Diff}^+_{S_n}(\Sigma_{0,n})/\mathrm{Diff}^\circ$$
$$= \text{diffeomorphisms permuting the n points}$$

is isomorphic to B_n/stuff, B_n the full n-stranded braid group.

The Dehn twist along the loop 

permutes $(i, i+1)$ and corresponds to the braid 

$= \sigma_i$

We have $B_n = \langle \sigma_1, \cdots, \sigma_{n-1} \rangle$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i - j| \geq 2$$

Recall the definition of $\widehat{GT}$

$$\widehat{GT} = \{(\lambda, f) \in \hat{\mathbb{Z}}^* \times \hat{F}_2' | (0) \begin{cases} x \mapsto x^\lambda \\ y \mapsto f^{-1} y^\lambda f \end{cases} \quad \text{induces an automorphism of } \hat{F}_2$$

$$(I) f(x,y) f(y,x) = 1$$

$$(II) f(x,y) x^m f(z,x) z^m f(y,z) y^m = 1,$$

$$m = \frac{\lambda - 1}{2},\ xyz = 1$$

$$(III) f(x_{34}, x_{45}) f(x_{51}, x_{12}) f(x_{23}, x_{34}) f(x_{45}, x_{51})$$

$$\cdot f(x_{12}, x_{23}) = 1 \text{ in } \hat{\Gamma}_{0,5}$$

We sketched 2 proofs of the fundamental result

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$$

Here is ******

then $M_{0,[n]}$ is also the definition over $\mathbb{Q}$ so a usual we have

$$G_{\mathbb{Q}} \to \mathrm{Out}(\hat{\pi}_1(M_{0,[n]})) = \mathrm{Out}(\hat{\Gamma}_{0,[n]})$$

Now $\pi_1(M_{0,[4]}) = \Gamma_{0,[4]} = \pi_1(M_{1,1}) \cong \mathrm{PSL}_2(\mathbb{Z})$ because
$M_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}/S_4 \xrightarrow{\sim} M_{1,1}$

$$\text{unordered } \{x_1, x_2, x_3, x_4\} \mapsto \text{ elliptic curve ramified over those points}$$

$$\{0, 1, \infty, \tau\} \mapsto y^2 = x(x-1)(x-\tau)$$

$$F_2 = \Gamma_{0,4} \hookrightarrow \Gamma_{0,[4]} = \langle \sigma_1, \sigma_2 \rangle$$
$$x, y \mapsto \sigma_1^2 \sigma_2^2$$

So (λ, f) acts on $\hat{\Gamma}_{0,[4]}$ by $\underbrace{\begin{array}{ll} \sigma_1 & \mapsto \sigma_1^\lambda \\ \sigma_2 & \mapsto f^{-1}\sigma_2^\lambda f \end{array}}_{*}$ And this is an automorphism

However $\begin{array}{ccc} \mathrm{PSL}_2(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}) & * & (\mathbb{Z}/3\mathbb{Z}) \\ \langle \sigma_1\sigma_2\sigma_2 \rangle & * & \langle \sigma_1\sigma_2 \rangle \end{array}$

$(*)$ easily shows that

$$\begin{aligned} (\lambda, f) : \sigma_1\sigma_2\sigma_1 &\mapsto \sigma_1\sigma_2\sigma_1 f(\sigma_1^2\sigma_2^2) \\ \sigma_1\sigma_2 &\mapsto \sigma_1\sigma_2(***)^2 f(\sigma_1^2\sigma_2^2) \end{aligned}$$

Then the relation $(\sigma_1\sigma_2\sigma_3)^2 = 1 \Rightarrow f(\sigma_2^2\sigma_1^2)f(\sigma_1^2\sigma_2^2) = 1$ and $(\sigma_1\sigma_2)^3 = 1 \Rightarrow$ $f(\sigma_1^2\sigma_2^2)\sigma_1^{\lambda-1}f(\sigma_3^2\sigma_1^2)\sigma_3^{\lambda-1}f(\sigma_2^2\sigma_3^2)\sigma_2^{\lambda-1} = 1$ (rel(II)).

Ihara showed that (III) is necessary and sufficient for (λ, f) to extend from an automorphism of $\hat{\Gamma}_{0,[4]}$ to one of $\hat{\Gamma}_{0,[5]}$ where

$$\hat{\Gamma}_{0,[4]} \subset \hat{\Gamma}_{0,[5]}$$
$$\langle \sigma_1, \sigma_2 \rangle \subset \langle \sigma_1, \sigma_2, \sigma_3, \sigma_3 \rangle$$

Now we see the true original TWO-LEVEL PRINCIPLE even though there are many relations in the groups $\hat{\Gamma}_{0,[n]}$ $n > 5$ we still have

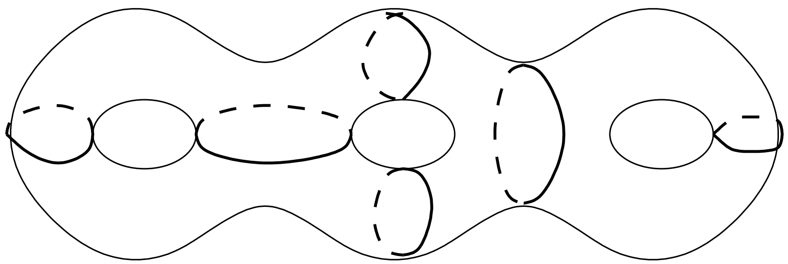
Theorem.

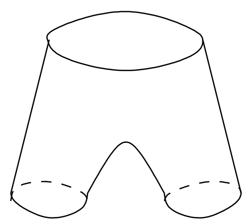
$$\widehat{GT} \hookrightarrow Aut(\hat{\Gamma}_{0,[n]}) \quad \forall n \geq 4$$
$$(\lambda, f) : \sigma_i \mapsto f(\sigma_i^2, y_i)\sigma_i^\lambda f(y_i, \sigma_i^2)$$

with $y_i = \sigma_{i-1} \cdots \sigma_1 \sigma_1 \cdots \sigma_{i-1}$

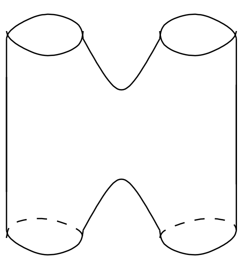
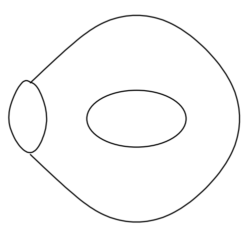
Pants decomposition

$\{2g-2+n$ simple closed loops disjoint on $\Sigma_{g,n}\}$

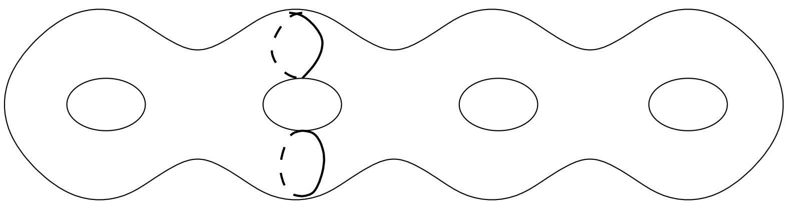


cuts the surface into "pants"  $\leftarrow$ smallest lego block $(0,3)$.

Removing any loop (erasing it) gives one larger Lego lock of type ether

 (0,4) or  (1,1)

Cutting loops can also be considered as “subsurface inclusion”



$\Sigma' \to \Sigma.$

This gives a morphism on the moduli spaces which is easy to see on their π_1s

$$\hat{\Gamma}(\Sigma') \mapsto \hat{\Gamma}(\Sigma)$$

$$\text{Dehn twist along } \gamma \mapsto \text{Dehn twist along} * * * *$$

for all simple closed loops γ supported on Σ'

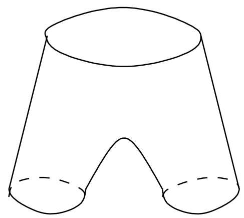
Theorem. $\widehat{GT}_g \to Out(\mathcal{C}_\pi)$ *where* $\mathcal{C}_\pi$ *is the category of* $\pi_1(M_{g,n}) = \hat{\Gamma}_{g,n}$ *with subsurface inclusion morphisms*

Remark. $\widehat{GT}_g$ *was defined with relations ensuring that* (λ, f) *acts on* $\Gamma_{0,4}$, $\Gamma_{0,5}$, $\Gamma_{1,1}$, $\Gamma_{1,2}$ *and the theorem shows that these two levels* $(3g-3+n=1,2)$ *suffices for* $\widehat{GT}_g$ *to be an automorphism *****

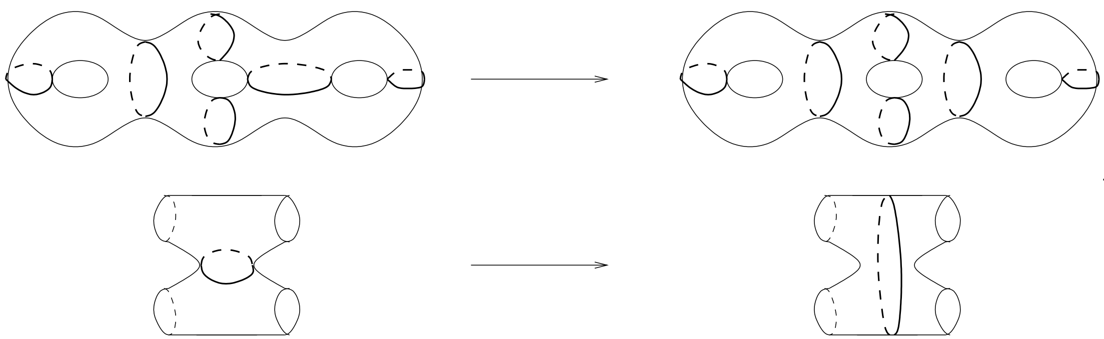
Recall that the action of $G_{\mathbb{Q}}$ on $\hat{\Gamma}_{g,n}$ (or any π_1) preserves (cyclic) inertia subgroups. Here inertia is given by Dehn twists d_γ along loops γ, so

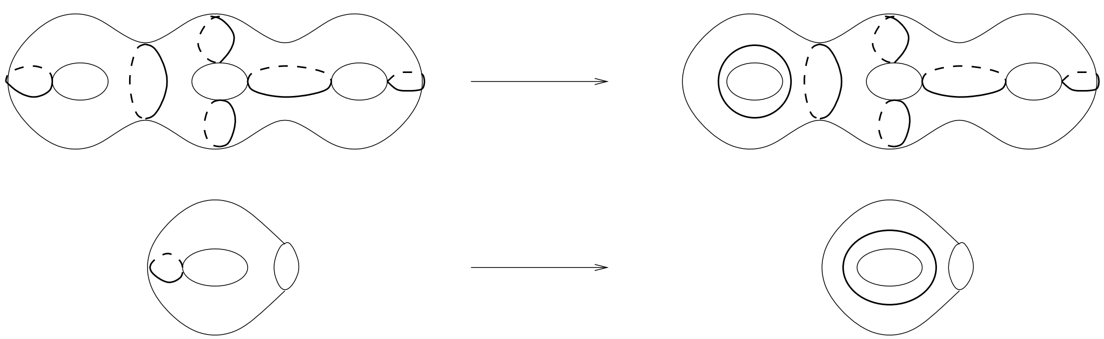
$$\sigma(d_\gamma) = F_\sigma^{-1} d_\gamma^{\chi(\sigma)} F_\sigma$$

$F_\sigma \in \hat{\Gamma}_{g,n}$ was mysterious but now thanks to $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}_g$ and the Lego-game property of $\widehat{GT}_g$ this $\hat{F}_\sigma$ can be explicitly computed.

Lego: building Σ with blocks  in different ways.

Generators of the game

1.  A-move

2.  S-move

These generate *** game

Theorem. *The $G_{\mathbb{Q}}$ and $\widehat{GT}_g$ actions on ****

Pick a pants decomposition P on Σ let $(\lambda, f) \in \widehat{GT}_g$ (assume $\lambda = 1$ $\rho_2 = 0$ for simplicity)

Then we associate $(\lambda, f) \mapsto F_p \in \mathrm{Aut}(\hat{\Gamma}_{g,n})$ with

(i) $F_p(d_\alpha) = d_\alpha \ \forall \alpha \in P$

(ii) $F_p(d_\beta) = f(d_\beta d_\alpha) d_\beta f(d_\alpha d_\beta)$ if $\alpha \mapsto \beta$ is an A move

(iii) $F_p(d_\beta) = f(d_\beta^2 d_\alpha^2) d_\beta f(d_\alpha^2 d_\beta^2)$ if $\alpha \mapsto \beta$ is an S move

Now, since we can obtain any loop γ on Σ by moving the loop of P around by A and S-moves we find

$$F_p(d_\gamma) = f(\ )^{-1} f(\ )^{-1} \cdots d_\gamma f(\ ) \cdots f(\cdots)$$

$$\downarrow\downarrow\downarrow\downarrow\downarrow$$

A and S moves

The only difficulty is a topological result showing that the conjugating quantity is independent of the choice of sequences of moves

Conclusion: Using GT theory greatly clarifies the action of $G_{\mathbb{Q}}$ on $\pi_1(M_{g,n})$ and reveals its surprising Lego-combinatorial nature.