RECALL:

$$G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^* \times \widehat{F_2}'$$

$$\sigma \mapsto (\chi(\sigma), f_\sigma)$$

Strategy to understand f_σ:

1. Compute it in quotients of $\widehat{F_2}$.
2. Find necessary and sufficient conditions on $f \in \widehat{F_2}'$. The former we know.

We will consider fundamental groups of all moduli spaces of curves

$$\mathcal{M}_{g,n}$$

where each point is an isomorphism class of Riemann surfaces of genus g with n distinct points.

Special case: $g = 0$

Let $(x_1, \ldots, x_n) \in \mathbb{P}^1$, then $(x_1, \ldots, x_n) \cong (y_1, \ldots, y_n)$ iff there exists

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C})$$

such that

$$\gamma(x_i) = \frac{ax_i + b}{cx_i + d} = y_i \qquad 1 \leq i \leq n$$

Example:

Given $(x_1, \dots, x_n)$ there exists a unique $\gamma \in PSL_2(\mathbb{C})$ such that

$$\gamma(x_1) = 0$$

$$\gamma(x_2) = 1$$

$$\gamma(x_3) = \infty$$

So there exists a unique representative in each isomorphism class of the form

$$(0, 1, \infty, y_1, \dots, y_{n-3})$$

So

$$\mathcal{M}_{0,n} = (\mathbb{P} - \{0, 1, \infty\})^{n-3} - \Delta$$

where $\Delta = \bigcup\{y_i = y_j\}$

$$\mathcal{M}_{0,4} = \mathbb{P} - \{0, 1, \infty\}$$

Idea for sufficient conditions:

Take all conditions on an $f \in \widehat{F_2}'$ coming from its actions on all π_1's of all $\mathbb{C}$ varieties.

Then f does come from $G_{\mathbb{Q}}$.

Category $\mathcal{C}$:

Objects: some $\mathbb{Q}$-varieties.

Morphisms: all $\mathbb{Q}$-morphisms φ

Associated category $\mathcal{C}_\pi$:

Objects: $\pi_1(X)$ for all $X \in \mathcal{C}$

Morphisms: φ_* up to inner automorphisms.

$$\mathrm{Aut}(\mathcal{C}_\pi) = \{(\phi_X)_{X\in\mathcal{C}} | \phi_X \in \mathrm{Aut}(\pi_1(X))\}$$

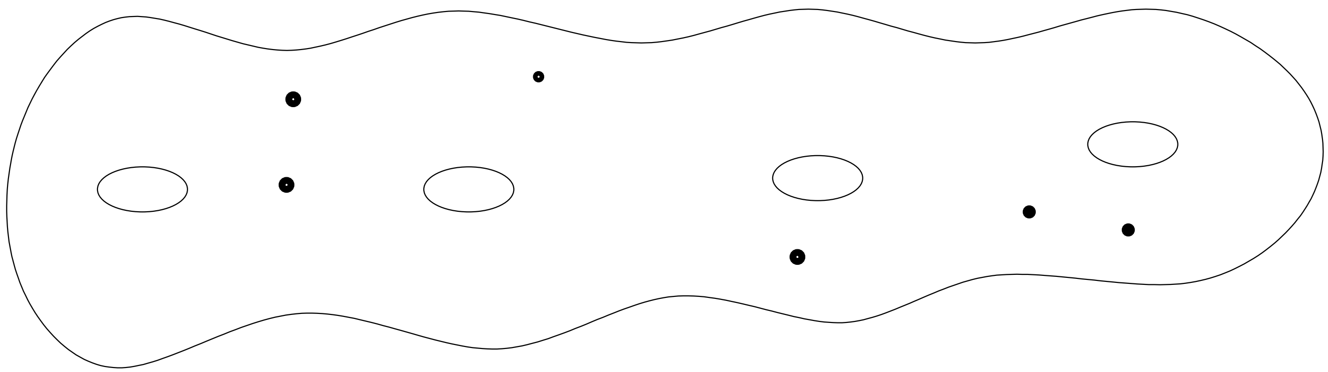
$$\begin{array}{ccc}
\pi_1(X) & \xrightarrow{\varphi_*} & \pi_1(Y) \\
\psi_X \downarrow & \circlearrowleft & \downarrow \phi_Y \\
\pi_1(X) & \xrightarrow{\varphi_*} & \pi_1(Y)
\end{array}$$

Pop's unpublished theorem answering a question of Oda-Matsumoto:

$$\mathrm{Aut}(\mathcal{C}_\pi) = G_{\mathbb{Q}}$$

if $\mathcal{C}$ ={all φ-varieties}

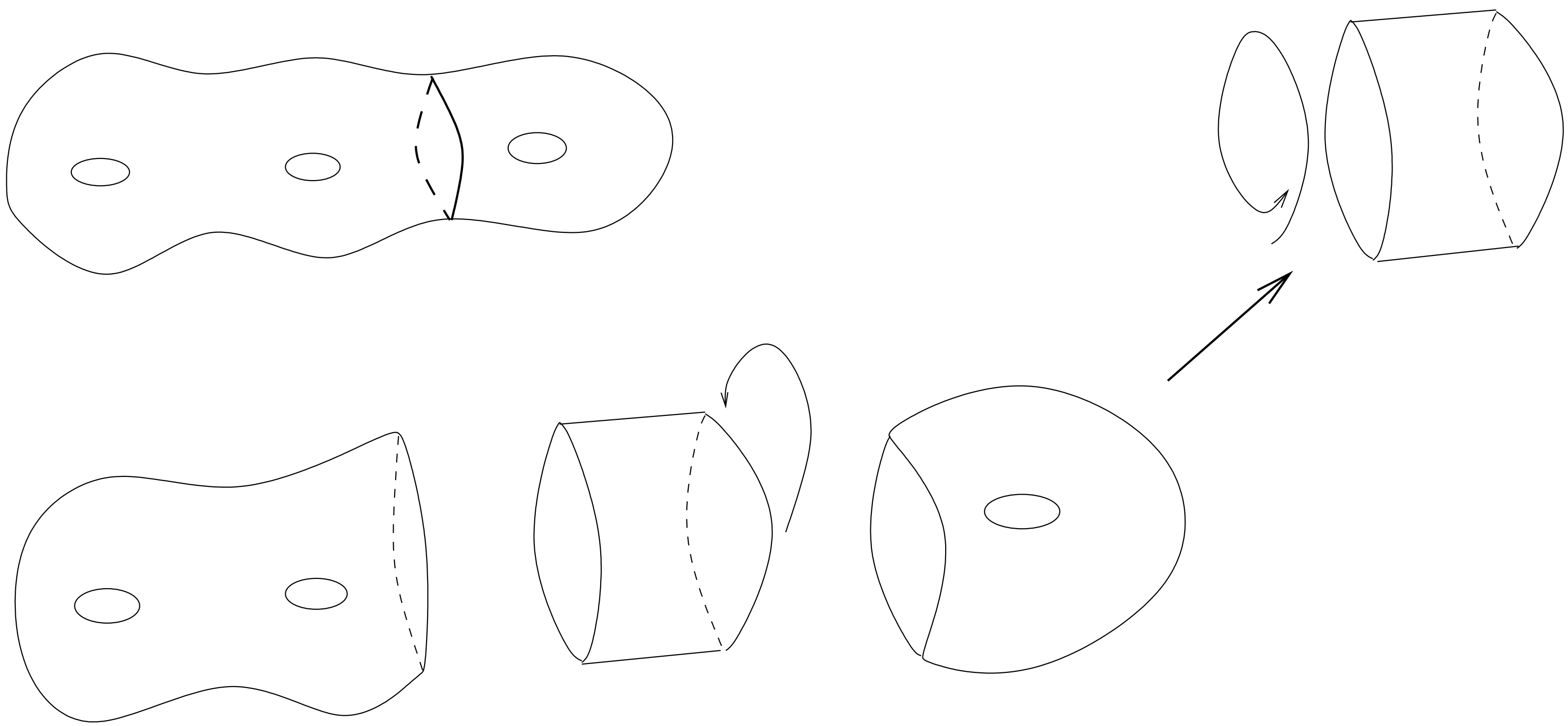
Let $\Sigma_{g,n}$ be a Riemann surface (equipped with an analytic structure):



$$\text{Diffeomorphism} \longrightarrow \text{loop on } \mathcal{M}_{g,n}$$

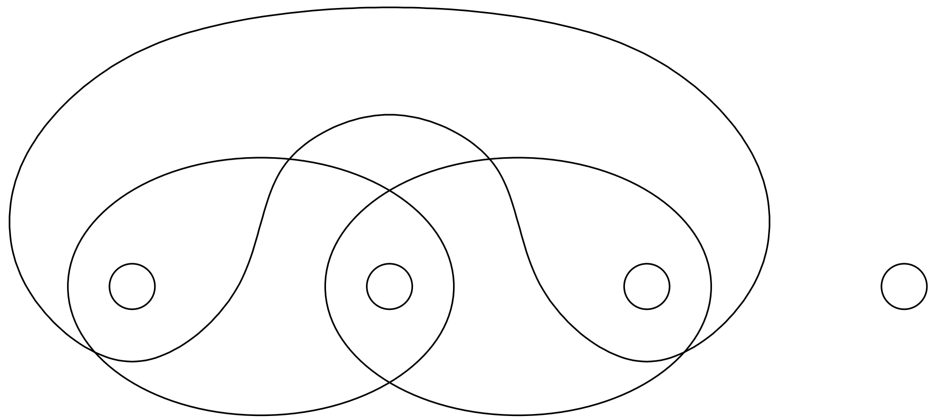
$$\mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^0(\Sigma) \xrightarrow{\sim} \pi_1(\mathcal{M}_{g,n})$$

A <u>generating set</u> of diffeos of Σ is the Dehn twists along simple closed loops on Σ



In genus 0:

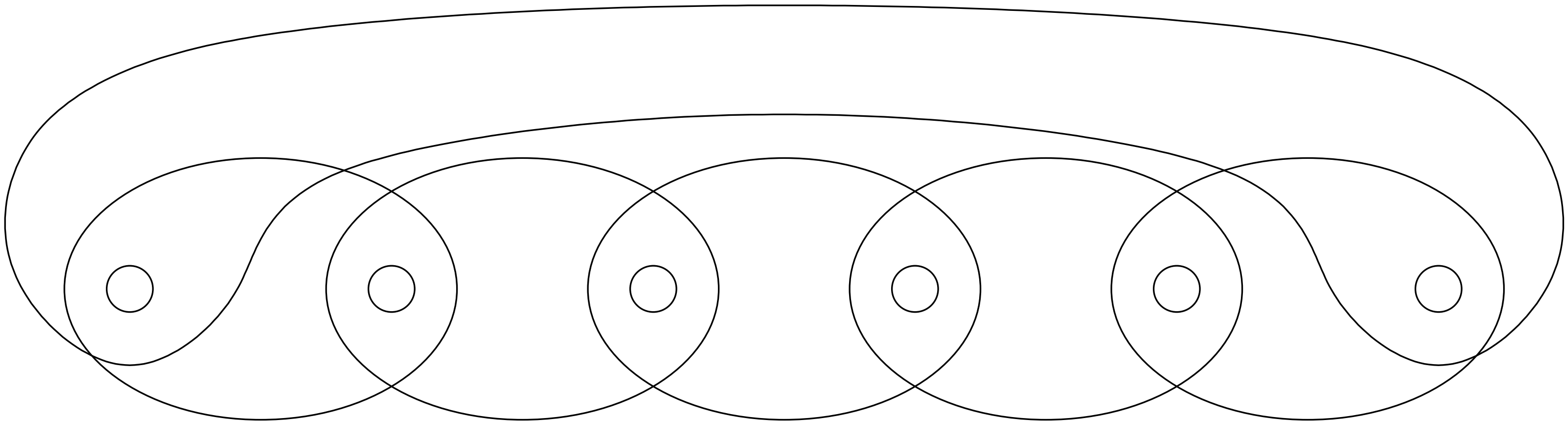
$\mathcal{M}_{0,4}$, $\Sigma_{0,4}$



where $x = x_{12}$, $y = x_{23}$, $z = x_{13}$, and $xyz = 1$

$$\begin{aligned}\pi_1(\mathcal{M}_{0,4}) &= \Gamma_{0,4} =< x, y, z | xyz = 1 > \\ &= F_2 = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\})\end{aligned}$$

$\mathcal{M}_{0,5}$



$$
\begin{aligned}
< x_{12}, x_{23}, x_{34}, x_{45}, x_{51} > \quad &= \quad \Gamma_{0,5} \\
&= \quad \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}^0(\Sigma) \\
&= \quad \pi_1(\mathcal{M}_{0,5})
\end{aligned}
$$

Notation:

$$x, y \mapsto a, b$$

$$\widehat{F_2} \to G$$

$$f \mapsto f(a, b)$$

Definition:(Grothendieck-Teicmüller group)

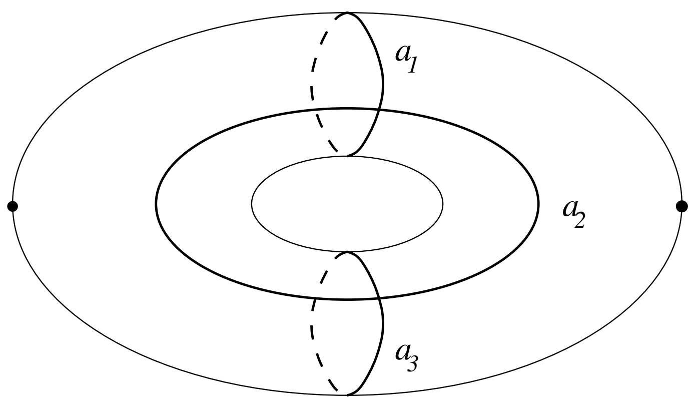
$$\widehat{GT} = \left\{ (\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F_2}' \middle| \begin{array}{c} (I) f(x,y) f(y,x) = 1 \\ (II) f(x,y) x^n f(z,x) z^n f(y,z) y^n = 1 \\ (III) f(x_{12}, x_{23}) f(x_{23}, x_{34}) f(x_{34}, x_{45}) f(x_{51}, x_{12}) = 1 \end{array} \right\}$$

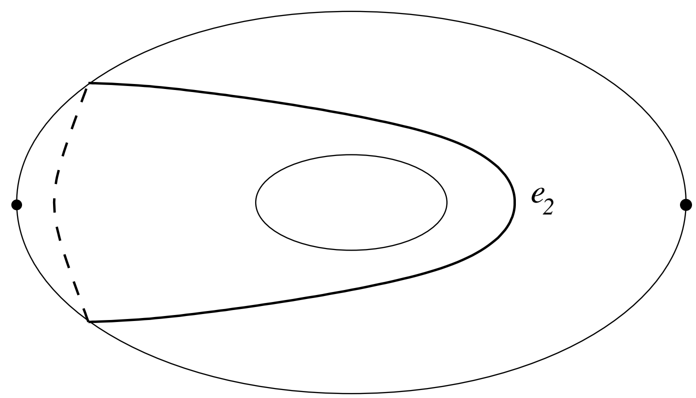
$$\widehat{GT}_g = \{(\lambda, f) \in GT | f(e_1, a_1) a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3 a_2 a_3)^{2m} f(e_2, e_1) e_2^{2m} f(e_3, e_2) a_2^{-2m}$$

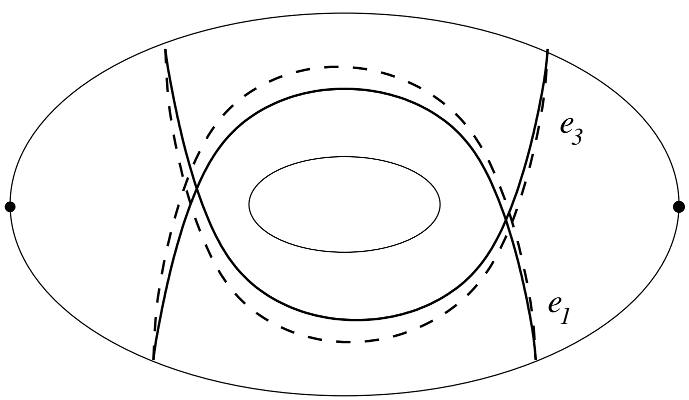
$$(a_1 a_2 a_1)^{2m} f(a_1^2, a_2^2) a_1^{8\rho_2} f(a_3, e_3) = 1\}$$

$$\exists g \in \widehat{F_2} \text{ such that } f(x,y) = g(y,x)^{-1} g(x,y)$$

$$\overline{g(x,y)} = (x,y)^{\rho_2} \in \widehat{F_2}^{\text{ab}}$$







Theorem 1:

$$G_{\mathbb{Q}} \hookrightarrow \widehat{GT}_g \hookrightarrow \widehat{GT}$$

Theorem 2:

$$\begin{aligned} \widehat{GT} &= \mathrm{Aut}(\mathcal{C}_\pi) \\ \mathcal{C} &= < \text{all genus zero } \mathcal{M}_{0,n} \text{ moduli spaces} > \\ &= < \mathcal{M}_{0,4}, \mathcal{M}_{0,5} > \text{(two-level principle)} \\ \dim \mathcal{M}_{g,n} &= 3g - 3 + n \end{aligned}$$

Theorem 3:

$$\begin{aligned} \widehat{GT}_g &= \mathrm{Aut}(\mathcal{C}_\pi) \\ \mathcal{C} &= < \text{all } \mathcal{M}_{g,n} > \\ &= < \mathcal{M}_{0,4}, \mathcal{M}_{0,5}, \mathcal{M}_{1,1}, \mathcal{M}_{1,2} > \end{aligned}$$

Ask: $G_{\mathbb{Q}} \overset{???}{=} \widehat{GT}_g$

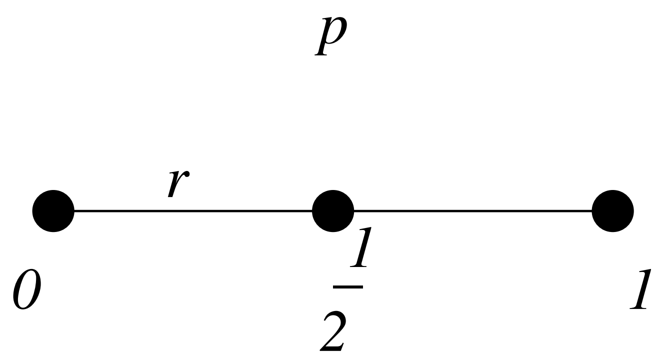
$$S_n \hookrightarrow \mathrm{Out}(\widehat{\Gamma_{0,n}}) \qquad n > 4$$

$$S_n = \mathrm{Aut}(\mathcal{M}_{0,n})$$

S_n induces an outer automorphism of $\widehat{\Gamma_{0,n}}$ by $x_{ij} \mapsto x_{\sigma(i)\sigma(j)}$

Theorem:

$$G_{\mathbb{Q}} \hookrightarrow \mathrm{Out}_{S_n}(\widehat{\Gamma_{0,n}}) = \widehat{GT}$$



$$\sigma(p) = pf_\sigma$$

$$\sigma(r) = rg_\sigma$$

$$\theta(z) = 1 - z$$

$$\theta(r)^{-1}r = p$$

$$\begin{aligned}
\theta(rg_r)^{-1}rg &= pf_\sigma \\
&= \theta(g_r)^{-1}g_\sigma \\
&= f_\sigma \Leftrightarrow g_\sigma(yx)^{-1}g_\sigma(xy) = f_\sigma(xy)
\end{aligned}$$