Linear Grothendieck-Teichmüller Theory

grt = graded "GT" Lie algebra

$grt = \{f \in \mathrm{Lie}[x,y]$ s.t.

(I) $f(x,y) + f(y,x) = 0$

(II) $f(x,y) + f(z,x) + f(y,z) = 0,\ x + y + z = 0$

(III) $f(x_{12}, x_{23}) + f(x_{23}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{51}) +$
$f(x_{51}, x_{12}) = 0\}$

where $x_{i,i+1}$ are generators of another Lie algebra (the 5-strand braids) generated by $x_{ij}, 1 \leq i, j \leq 5$ with relations

$$x_{ii} = 0, x_{ij} = x_{ji}$$

$$\sum_{i=1}^{5} x_{ij} = 0$$

$$[x_{ij}, x_{ik}] + [x_{ji}, x_{jk}] + [x_{ki}, x_{kj}] = 0.$$

Multizeta Values

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \ k_i \in \mathbb{Z}$$

$$\in \mathbb{R} \text{ when } k_1 \geq 2$$

Thm. The MZV's form a $\mathbb{Q}$-algebra

Proof 1:

$\zeta(k_1, \ldots, k_r) =$

$$(-1)^r \int_0^1 \frac{dt_n}{t_n - \epsilon_n} \int_0^{t_n} \frac{dt_{n-1}}{t_{n-1} - \epsilon_{n-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - \epsilon_1}$$

$$(\underbrace{0, \ldots, 0}_{k_1-1}, 1, \underbrace{0, \ldots, 0}_{k_2-1}, 1, \ldots, \underbrace{0, \ldots, 0}_{k_r-1}, 1) = (\epsilon_n, \ldots, \epsilon_1)$$

$$x^{k_1-1}y \cdots x^{k_r-1}y$$

$$\zeta(2) = \int_0^1 \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} \qquad (0,1) = (\epsilon_2, \epsilon_1)$$

$$= \int_0^1 \frac{dt_2}{t_2} \int_0^{t_2} \sum_{n \geq 0} t_1^n dt_1$$

$$= \sum_{n \geq 0} \frac{1}{n+1} \int_0^1 \frac{dt_2}{t_2} [t_1^{n+1}]_0^{t_2}$$

$$= \sum_{n \geq 0} \frac{1}{n+1} \int_0^1 t_2^n dt_2 = \sum_{n \geq 0} \frac{1}{(n+1)^2} [t_2^{n+1}]_0^1 = \sum_{n \geq 1} \frac{1}{n^2}$$

$$(-1)^r \int_{0\le t_1\le\cdots\le t_r\le 1} \omega_1\cdots\omega_r \int_{0\le s_1\le\cdots\le s_n\le 1} \gamma_1\cdots\gamma_n =$$

$$\sum_{\text{shuffles}} \int_{\text{standard simplex}}$$

$$\int_0^1 \int_0^{t_2} \frac{dt_2}{t_2}\frac{dt_1}{1-t_1} \cdot \int_0^1 \frac{ds_3}{s_3}\int_0^{s_3} \frac{ds_2}{s_2-1}\int_0^{s_2} \frac{ds_1}{s_1-1}$$

$$(0 \le t_1 \le t_2 \le s_1 \le s_2 \le s_3 \le 1)$$

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10 simplices whose union =

$(0 \le t_1 \le t_2 \le 1) \times (0 \le s_1 \le s_2 \le s_3 \le 1)$

$$\sum_{10 \text{ simplices}} = \int \frac{dt_1}{1-t_1} \frac{dt_2}{t_2} \frac{ds_1}{s_1-1} \frac{ds_2}{s_2-1} \frac{ds_3}{s_3} +$$

$$\int \frac{dt_1}{1-t_1} \frac{ds_1}{s_1-1} \frac{dt_2}{t_2} \frac{ds_2}{s_2-1} \frac{ds_3}{s_3} + \cdots$$

$$\zeta(k_1,\ldots,k_r)=\zeta(w) \qquad w=x^{k_1-1}y\cdots x^{k_r-1}y$$

$$\zeta(w)\zeta(v)=\sum_{u\in \mathrm{sh}(w,v)}\zeta(u)$$

Proof 2:

$$\zeta(k_1,\ldots,k_r)\zeta(l_1,\ldots,l_s)$$

$$= \sum_{n_1>\cdots>n_r>0} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \cdot \sum_{m_1>\cdots>m_s>0} \frac{1}{m_1^{l_1}\cdots m_s^{l_s}}$$

$$= \sum_{n_1,\ldots,m_s} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}\cdots m_s^{l_S}}$$

$$\zeta(a)\zeta(b,c) = \sum_{n\geq 1}\frac{1}{n^a}\sum_{m_1>m_2>0}\frac{1}{m_1^b m_2^c}$$

$$= \sum_{n>m_1>m_2>0}\frac{1}{n^a m_1^b m_2^c} + \sum_{m_1>n>m_2>0}\frac{1}{m_1^b n^a m_2^c} +$$

$$\sum_{m_1>m_2>n>0}\frac{1}{m_1^b m_2^c n^a} + \sum_{n=m_1>m_2>0}\frac{1}{n^a n^b m_2^c} +$$

$$\sum_{m_1>n=m_2>0}\frac{1}{m_1^b n^a n^c}$$

$$= \zeta(a,b,c) + \zeta(b,a,c) + \zeta(b,c,a) + \zeta(a+b,c) + \zeta(a,b+c)$$

stuffle of two sequences

$\mathrm{st}((k_1,\ldots,k_r),(l_1,\ldots,l_s)) = \{$all shuffles$\} \cup \{$all shorter sequences obtained from the shuffles by adding "neighboring components" from different sequences$\}$

3 Main Conjectures on MZV's

1) They are all transcendent

1') There exist no linear relations between MZV's of different weight

$$\zeta(k_1,\ldots,k_r)^m + a_1\zeta(k_1,\ldots,k_r)^{m-1} + \cdots + a_0 = 0$$

$\Rightarrow$ linear relation in different weights

2) The only alg. relations between MZV's (in given weight) come from the 2 families

$$\zeta(u)\zeta(v) = \sum_{w \in \mathrm{sh}(u,v)} \zeta(w)$$

$$\zeta(k_1, \ldots, k_r)\zeta(l_1, \ldots, l_s) = \sum_{\underline{s} \in \mathrm{st}(\underline{k},\underline{l})} \zeta(\underline{s})$$

$\zeta(3) = \zeta(2,1) \ \rightarrow$ old

$\zeta(n) = \zeta(2,1,\ldots,1)\ \forall n$ antipode

$$\sum_{i_1+\cdots+i_k=n, i_1>1} \zeta(i_1,\ldots,i_k) = \zeta(n)$$

$$\zeta(\underbrace{3,1}_{n\text{ times}},\ldots,3,1) = \frac{1}{2n+1}\zeta(2,\ldots,2)$$

Def: Drinfel'd associator

I now want to extend the definition of $\zeta(w)$ (w word in x, y) from words xvy to all words in such a way that

$$\zeta(u)\zeta(v) = \sum_{w \in \mathrm{sh}(u,v)} \zeta(w) \quad \forall u, v$$

Drinfel'd $\to$ Le, Murakami

$\to$ Furusho

$w = y^a v x^b$ where v starts in x, ends in y

$$\zeta(w) = \sum_{r=0}^{a} \sum_{s=0}^{b} (-1)^{r+s} \sum_{u \in \mathrm{sh}(y^r, y^{a-r} v x^{b-s}, x^s)} \zeta(\pi(u))$$

$$\zeta(\emptyset) = 0$$

$$\pi(u) = \begin{cases} u & \text{if } u \text{ starts in } x \text{ ends in } y \\ 0 & \text{otherwise} \end{cases}$$

$\zeta(w) = \zeta(k_1, \ldots, k_r)$
$w = x^{k_1-1}y \cdots x^{k_r-1}y$ (convergent word)

Drinfel'd associator $\Phi_{KZ} = \sum_w (-1)^{d(w)}\zeta(w)w$
$d(w) =$ number of $y's$ in w

Consider $NZ = Z/\langle Z_0, Z_2, (Z_{\geq 0})^2\rangle$

$Z_0 = \mathbb{Q}, Z_1 = 0, Z_n = \langle\text{weight } n \text{ mult. zetas}\rangle$

$$Z = \bigoplus_{n\geq 0} Z_n \twoheadrightarrow \text{MZV's}$$

NZ is a vector space, graded by the weight, in weight ≥ 3

Φ = image of Φ_{KZ} in NZ

Theorem: $\Phi = \sum_w (-1)^{d(w)} \bar{\zeta}(w) w,\ \bar{\zeta}(w) \in NZ$ is in grt

e.g. $\Phi(x,y) + \Phi(y,x) = 0$ yields new relations on ζ's