L. notes p. 27 wrong

Recall: $$\mathrm{G}_{\mathbb{Q}} \longrightarrow Aut\pi \longrightarrow Aut\mathcal{L}$$

$$\qquad\qquad \| $$

$$\mathbb{Q}\langle\langle \underset{\log x}{\overset{}{\underset{\uparrow}{X}}}, \underset{\log y}{\underset{\uparrow}{Y}} \rangle\rangle$$

$$\rho_{\overrightarrow{01}} : X = \log x \to \log x^{\chi(\sigma)} = \chi(\sigma)X$$

§5 Cohomology

§5.1 Extension

$$0 \longrightarrow B \longrightarrow E \longrightarrow \mathbb{Q}_l \longrightarrow 0 \quad : \text{exact } \mathrm{G}_{\mathbb{Q}}\text{-modules}$$

$$e \longmapsto \overset{\text{trivial}}{1}$$

$\mathrm{Map}(\mathrm{G}_{\mathbb{Q}}, B) \ni [G \mapsto G(e) - e]$ is a 1-cocycle

$\Rightarrow \ [E] \in H^1(\mathrm{G}_{\mathbb{Q}}, B)$

$$\mathrm{G}_{\mathbb{Q}} \overset{\rho_{\overrightarrow{01}}}{\to} Aut\mathcal{L} \subset \underbrace{\mathbb{Q}\langle\langle \underset{\log x}{\underset{\uparrow}{X}} , \underset{\log y}{\underset{\uparrow}{Y}} \rangle\rangle}_{\in \pi^{(l)}}$$

$$X \overset{\sigma}{\to} \chi_l(\sigma) X$$

$$\qquad\qquad \in \mathcal{L}$$
$$\qquad\qquad \downarrow$$
$$Y \to \chi_l(\sigma) f_\sigma^{-1} Y f_\sigma$$
$$\uparrow$$
$$\in A^\times = \mathbb{Q}_l\langle\langle X, Y\rangle\rangle^\times$$

$$e \mapsto 1$$

$$0 \to B \to E \to \mathbb{Q}_l \to 0$$

$$||$$

trivial $\mathrm{G}_{\mathbb{Q}}$−module

:$\mathrm{G}_{\mathbb{Q}}$−mod

$$[E] \in Ext^1_{\mathrm{G}_{\mathbb{Q}}}(\mathbb{Q}_l, B)$$

$$||$$

$$H^1(G, B) = \text{cocycle}$$

$$\mathrm{G}_{\mathbb{Q}} \to B$$

$$\sigma \mapsto \sigma(e) - e$$

§5-2 Soulé's Cocycle

$$H^1(\mathrm{G}_{\mathbb{Q}}, \mathbb{Q}_l(m)) = \begin{cases} 0 & m \geq 2 \text{ even} \\ \mathbb{Q}_l & m \geq 3 \text{ odd} \end{cases}$$

$\exists \underset{m \geq 3 \text{ odd}}{\chi_m} : \mathrm{G}_{\mathbb{Q}} \to \mathbb{Q}_l(m)$ called Soulé's cocycle

Definition: OMIT

$$0 \to \mathbb{Q}_l(m) \to E_m \to \mathbb{Q}_l \to 0$$

$$H^1(\mathrm{G}_{\mathbb{Q}}, \mathbb{Q}_l(m)) = \begin{cases} 0 & m : \text{even} \geq 2 \\ \mathbb{Q}_l & m : \text{odd} \geq 3 \end{cases}$$

Soulé's Cocycle:

$\chi_m : \mathrm{G}_{\mathbb{Q}} \to \mathbb{Q}_l(m)$

$0 \to \mathbb{Q}_l(m) \to E_m \to \mathbb{Q}_0 \to 0$ (Unique nontrivial extension up to mult. by $\mathbb{Q}_l$)

§5.2 Extension from π

Def'n: $\mathbb{Q}_l(m) = \mathbb{Q}_l$ but $G_{\mathbb{Q}}$ acts by $\chi(\sigma)^m$ $(m \in \mathbb{Z})$ "m^{th} Tate module"

$$0 \to \mathcal{H} \to \mathcal{L} \to \mathbb{Q}\overline{X} \to 0$$

$$0 \to \mathcal{L}' \to \mathcal{H} \to \underset{\underset{\mathbb{Q}(1)}{\wr|}}{\mathbb{Q}\overline{Y}} \to 0$$

X

$\mathcal{H} \downarrow Y$

$$0 \to \underset{=\mathcal{L}(-1)}{\mathcal{L}' \otimes \mathbb{Q}_l(-1)} \to \underset{Y\otimes(-1)}{\mathcal{H} \otimes \mathbb{Q}_l(-1)} \to \underset{\overline{Y}\otimes 1_{-1}}{\mathbb{Q}_l\overline{Y} \otimes \mathbb{Q}_l(-1)} \to 0$$

$\mathcal{L}' \downarrow [X, Y]$

$[X, [X, Y]]$

$\mathcal{H}^2 \ni [Y, [X, Y]]$

$$\to [\pi_1] \in H^1(G_{\mathbb{Q}}, \mathcal{L}(-1))$$

$\vdots$

As a cocyle, $[\pi_1]$ is: $\sigma(Y \otimes 1_{-1}) - Y \otimes 1_{-1}$

$\mathcal{H}^3 \ni [Y, [Y, [X, Y]]]$

$$= \chi(\sigma) \cdot f_\sigma^{-1} Y f_\sigma \otimes \chi(\sigma)^{-1} 1_{-1} - Y \otimes 1_{-1}$$

$\overset{\cap}{\mathcal{H}^2}$

$$= (f_\sigma^{-1} Y f_\sigma - Y) \otimes 1_{-1}$$

X $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mathcal{L}$

$\downarrow \mathcal{H}^1 \quad Y$

$[X, Y] \quad \downarrow \mathcal{L}'$

$$0 \to \mathcal{H}^1 \to \mathcal{L} \to \mathbb{Q}_l X \to 0$$

$[X, [X, Y]]$

$$0 \to \underset{\underset{[\mathcal{L},\mathcal{L}]}{||}}{\mathcal{L}'} \to \mathcal{H}^1 \to \underset{\underset{\mathbb{Q}_l(1)}{|\wr}}{\mathbb{Q}_l Y} \to 0 \quad (*)$$

Def. $\mathbb{Q}_l(m) \simeq \mathbb{Q}_l \cdot 1_{(m)}$

$\mathrm{G}_{\mathbb{Q}} \ni \sigma \curvearrowright \chi_l(\sigma)^m \cdot 1_{(m)}$

$$G_{\mathbb{Q}} \curvearrowright M \qquad M(m) := M \otimes_{\mathbb{Q}_l} \mathbb{Q}_l(m)$$

$$\circledast \otimes (-1)$$

$$0 \to \mathcal{L}'(-1) \to \mathcal{H}^1(-1) \to \underset{\substack{\mathbb{Q}_l \\ \wr|}}{\mathbb{Q}_l(Y)}(-1) \to 0$$

$$Y \otimes 1_{(-1)} \mapsto Y \otimes 1_{(-1)}$$

$$\sigma : \sigma(Y \otimes 1_{(-1)}) - Y \otimes 1_{(-1)} \longrightarrow \cancel{\chi(\sigma)} f_\sigma^{-1} Y f_\sigma \otimes \cancel{\chi(\sigma)^{-1}} 1_{(-1)}$$

$$\longmapsto (f_\sigma^{-1} Y f_\sigma - Y) \otimes 1_{-1}$$

5.3 Soulé's Cocycle

quotient of $(*)$

$$0 \to \mathcal{L}'/(\mathcal{H}^3 + \mathcal{L}'')\ (-1) \to \mathcal{H}/(\mathcal{H}^3 + \mathcal{L}'')\ (-1) \to \mathbb{Q}_l \overline{Y}(-1) \to 0 \qquad (*/\sim)$$

$\uparrow$

Basis: $[X,Y], [X,[X,Y]], [X,[X,[X,Y]]], \cdots$

$[Y,[X,Y]], [Y,[X,[X,Y]]], \cdots$

$\wr$ $\mathbb{Q}_l(3)$ $\qquad$ $\wr$ $\mathbb{Q}_l(4)$

$\sigma : [X,Y] \mapsto [X,[F_\sigma, Y]]$

$P := \mathcal{H}^2/(\mathcal{H}^3 + \mathcal{L}'') \simeq \bigoplus_{m \geq 3} \mathbb{Q}_l(m)$

$G_{\mathbb{Q}}$-mod $\rightsquigarrow$ $\in H^1(G_{\mathbb{Q}}, P \otimes (-1))$

$\because$) cocycle for $(*/\sim)$ is

$$\sigma \mapsto (f_\sigma^{-1} Y f_\sigma - Y) \otimes 1_{(-1)} \subset \mathcal{L}'/(\mathcal{H}^3 + \mathcal{L}'')\ (-1) \qquad (F_\sigma := \log f_\sigma)$$

$$|| \quad \exp(-ad(F_\sigma) \cdot Y) \otimes 1_{(-1)}$$

$$(Y - [F_\sigma, Y] + \tfrac{1}{2!}[F_\sigma,[F_\sigma,Y]] - \cdots - Y) \otimes 1_{(-1)}$$

$|||$ mod $\mathcal{L}'$

$$[Y, F_\sigma] \otimes 1_{(-1)} \subset \mathcal{H}^2/(\mathcal{H}^3 + \mathcal{L}'')(-1)$$

$(*/\sim)$

$$0 \to \mathcal{L}'/(\mathcal{H}^3 + \mathcal{L}'')(-1) \to \mathcal{H}/(\mathcal{H}^3 + \mathcal{L}'')(-1) \to \mathbb{Q}_l Y(-1) \to 0$$

$$\bigcup$$

$$P := \mathcal{H}/(\mathcal{H}^3 + \mathcal{L}'') \simeq \prod_{m \geq 3} \mathbb{Q}_l(m) \qquad (\mathrm{G}_{\mathbb{Q}}\text{-mod.})$$

$\underline{\text{cocycle}}$: $\sigma \mapsto (f_\sigma^{-1} Y f_\sigma - Y) \otimes 1_{(-1)} \bmod (\sim)$

$\underset{\in \mathcal{L}}{F_\sigma} := \log(f_\sigma)$

$$\begin{aligned}
\sigma(Y) - Y &= f_\sigma^{-1} Y f_\sigma - Y \\
&= \exp(ad(-F_\sigma)) \cdot Y \\
&= Y - [F_\sigma, Y] + \underbrace{\frac{1}{2!}[F_\sigma, [F_\sigma, Y]] + \cdots}_{\in \mathcal{L}''} - Y \qquad \bmod(\sim) \\
&\Rightarrow [Y, F_\sigma]
\end{aligned}$$

$K : \sigma \mapsto [Y, F_\sigma] \otimes 1_{(-1)} \in P$

So if $F_\sigma \underset{\mathcal{H}^2+\mathcal{L}'}{\equiv} : \sum_{m\geq 2} \frac{c_{m-1,1}(\sigma)}{(m-1)!} \underbrace{\underbrace{[X,[X,\cdots[X,Y]}_{m}\cdots]]}_{:=V_m} \mod \mathcal{H}^2$

its m^{th} component is

$K_m : \sigma \mapsto \frac{c_{m-1,1}(\sigma)}{(m-1)!}[Y, V_m] \otimes (-1) \in \mathbb{Q}_l(m)$

Th. Anderson Coleman Ihara

$$c_{m-1,1}(\sigma) = (1 - l^{m-1})\chi_m \sigma$$

$$\chi_m(\sigma) :$$

$$\zeta_M^{\chi_m(\sigma)} \underset{\sim}{\longleftarrow} \prod_{a \in (\mathbb{Z}/l^n)^\times} \left(\frac{\sigma\left((\sigma^{-1}(\zeta_{l^n}^a) - 1)^{\frac{1}{M}}\right)}{\left(\zeta_{l^n}^a - 1\right)^{\frac{1}{M}}} \right)^{a^{m-1}}$$

$$K : \sigma \mapsto [Y, \underset{\in \mathcal{L}'}{F_\sigma}] \otimes 1_{(-1)} \in \mathcal{H}^2 / \sim = P \simeq \prod \mathbb{Q}_l(m)$$

So F_σ must have at least 1 Y.

$$F_\sigma \underset{\mathcal{H}^2 + \mathcal{L}''}{\equiv} \sum \tfrac{c_{m-1,1}(\sigma)}{(m-1)!} \underbrace{[X, [X, \cdots [X, Y]}_{\text{length } m} \cdots]]_{:= V_m}$$

$$K_m : \sigma \mapsto \tfrac{c_{m-1,1}(\sigma)}{(m-1)!} [Y, V_m] \otimes 1_{(-1)}$$

$$\in H^1(\mathrm{G}_{\mathbb{Q}}, \mathbb{Q}_l(m)) \qquad P_{(-1)} \twoheadrightarrow \mathbb{Q}_l(m+1)(-1)$$

Th. $c_{m-1,1}(\sigma) = (1 - l^{m-1})^{-1}\chi_m(\sigma)$