$\chi(\sigma) \in \hat{\mathbb{Z}}^{\times}$

$f_\sigma \in \hat{F}_2'$

3-2 $f_\sigma \bmod \underline{\underline{N}} = [\hat{F}_2, \hat{F}_2]$

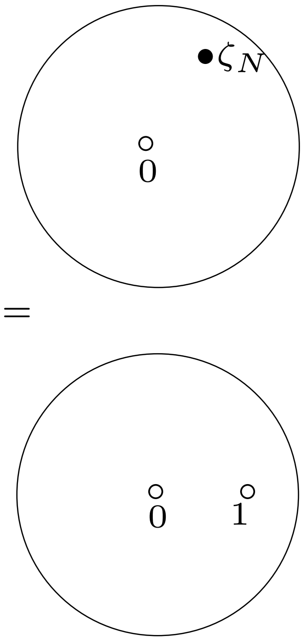
$\underline{\underline{N}} := H'$

$$1 \longrightarrow H_N \longrightarrow \hat{F}_2 \longrightarrow \overset{\text{additive}}{\mathbb{Z}/N} \longrightarrow 0$$

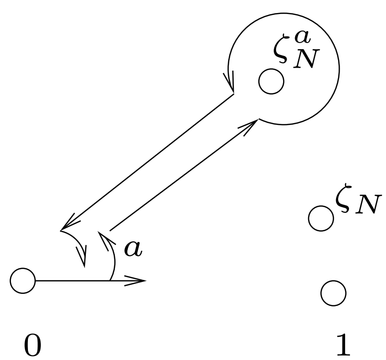
$$\cup$$

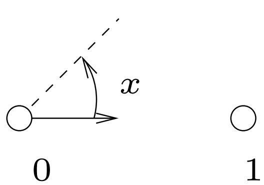
$$\langle x^N, y, x^{-a} y x^a (a \in \mathbb{Z}/N) \rangle$$

covering group $=$
$\mathbb{Z}/N$



$$X^N \longrightarrow X \qquad z \mapsto t = z^N \qquad \cdots \qquad H_N \to F_2 \qquad x^{-a}yx^a$$





$$H := \bigcap_{N \in \mathbb{N}} \hat{H}_N = Ker(\hat{F}_2 \underset{\substack{y \mapsto 0 \\ x \mapsto 1}}{\longrightarrow} \hat{\mathbb{Z}})$$

$$\cup$$

$$\hat{F}_2' \qquad\qquad z = t^{\frac{1}{N}}$$

$$\overline{f_\sigma} \in \hat{H}_N / \hat{H}'_N = \pi_1(\overline{X}_N, \overrightarrow{01})^{ab}$$

$$\simeq \mathrm{Gal}(\overline{\mathbb{Q}}(\{z^{\frac{1}{M}}, (\underbrace{\zeta_N^a - z}_{= u_a})^{\frac{1}{M}} | M \in \mathbb{N},\ a \in \mathbb{Z}/N\}) / \overline{\mathbb{Q}}(z))$$

$$u_a = (\zeta_N^a - t^{\frac{1}{N}}) \in \overline{\mathbb{Q}}(t^{\frac{1}{N}})$$

$$x^{-b}yx^b(u_a^{\frac{1}{M}}) = \begin{cases} \zeta_M u_a^{\frac{1}{M}} & a = b \\ u_a^{\frac{1}{M}} & a \neq b \end{cases}$$

$$\underset{\in H_N/H'_N}{\overline{f_\sigma}} : t^{\frac{1}{NM}} \mapsto t^{\frac{1}{NM}}$$

$$u_a^{\frac{1}{M}} \mapsto \zeta_M^{\overbrace{\kappa_{N,a(\sigma),M}}^{\in \mathbb{Z}/M}} u_a^{\frac{1}{M}}$$

$$\varprojlim_M \underset{\in \mathbb{Z}/M}{\kappa_{N,a(\sigma),M}} := \underset{\in \hat{\mathbb{Z}}}{\kappa_{N,a(\sigma)}}$$

$$f_\sigma \bmod H'_N = \prod_{a \in \mathbb{Z}/N} (x^{-a} y x^a)^{\kappa_{N,a(\sigma)}}$$

$$\hat{G} \ni g$$

$$\hat{\mathbb{Z}} \ni \hat{n} \Rightarrow g^{\hat{n}}$$

$$\zeta_M^{\kappa_{N,a(\sigma),M}} = \sigma(\sigma^{-1}(\zeta_N^a) - 1)^{\frac{1}{M}} (\zeta_N^a - 1)^{-\frac{1}{M}}$$

Corollary: $f_\sigma \bmod H' = \varprojlim_{N \to \infty} \prod_{a \in \mathbb{Z}/N} (x^{-a} y x^a)^{\kappa_{N,a(\sigma)}}$.

$$\varprojlim H_N / H'_N = H/H'$$

4. Lie alg-tion of $\pi_1 = F_2$

4-1 Malcev compl. F_2 OMIT

$$\text{———}'' \qquad F_2^{(l)} := \varprojlim_{[\hat{F}_2 : \underset{\text{open}}{N}] = l^a} \hat{F}_2/N \twoheadleftarrow \hat{F}_2$$

$$\mathrm{G}_{\mathbb{Q}} \hookrightarrow Aut\ \hat{F}_2 \longrightarrow Aut\ F_2^{(l)}$$

$\pi := F_2^{(l)} \ni x, y$

Def: $A := \mathbb{Q}_{(l)}\langle\langle X, Y\rangle\rangle$

$$= \{a_\emptyset 1 + a_X X + a_Y Y + a_{XY} XY + a_{YX} YX\}$$

Def: $\pi \xhookrightarrow[\text{Lazard}]{\text{gp. hom}} A^x$

$$x \longmapsto \exp(X) \longleftarrow \text{cont} => \text{ext}$$

$$y \longmapsto \exp(Y)$$

$$x^{l^n} \xrightarrow[n\to\infty]{} 1$$

$$\exp(X+Y) \neq \exp(X)\cdot\exp(Y)$$

$$\underset{Y,X\ \text{comm.}}{=}$$

$$\exp(X)^n = \exp(nX) \quad n \in \mathbb{Z}$$

$$x^{l^n} \longmapsto \exp(X)^{l^n} = \exp(l^n X) \xrightarrow[l^n \to 0]{} 1 \qquad (\text{in } \mathbb{Q}_l)$$

$$
\begin{array}{ccccc}
\pi & \hookrightarrow & \mathcal{G} & \xleftarrow[\text{completion}]{\text{Malcev}} & A^{\times} \\
 & & \Vert & & \\
 & & (\text{gp.})\,\exp(\mathcal{L}) & &
\end{array}
$$

$\mathcal{L} = \mathbb{Q}_l$ Lie algebra generated by X, Y completed $\qquad [X,Y] = XY - YX$

$$\overbrace{a_X}^{\in\mathbb{Q}_l} X + a_Y Y + a_{[X,Y]}[X,Y] + a_{[Y,[X,Y]]}[Y,[X,Y]]$$

$$\mathrm{G}_{\mathbb{Q}} \longrightarrow Aut\ F_2^{(l)} \longrightarrow \underset{\text{group}}{Aut\ \mathcal{G}} \xrightarrow{\sim} \underset{\text{Lie}}{Aut\ \mathcal{L}}$$

$$\chi_l : \mathrm{G}_{\mathbb{Q}} \to \hat{\mathbb{Z}}^\times = \prod \mathbb{Z}_l^\times \underset{\text{pr.}}{\to} \mathbb{Z}_l^\times$$

Lie alg-tion of Gal. Rep.

$$\begin{aligned} s_{\overrightarrow{01}}(\sigma) : X := \log x \quad &\mapsto \log(x^{\chi(\sigma)}) &&= \chi_l(\sigma)\log(x) = \chi_l(\sigma)X \\ Y := \log y \quad &\mapsto \log(f_\sigma^{-1} y^{\chi_l(\sigma)} f_\sigma) &&= f_\sigma \underset{\in\mathbb{Q}_l}{\chi_l(\sigma)} \log(y) f_\sigma^{-1} = \chi_l(\sigma) f_\sigma^{-1} Y f_\sigma \end{aligned}$$