- $K = \mathbb{Q}$ (Assume)

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{X}, \overrightarrow{01}) & \longrightarrow & \pi_1(X, \overrightarrow{01}) & \longrightarrow & G(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow & 1 \\
 & & \| & & \| & \overset{\exists s_x}{\curvearrowleft} & \| & & \\
 & & \mathrm{Gal}\big(M_{\overrightarrow{01}}/\overline{\mathbb{Q}}(t)\big) & & G\big(M_{\overrightarrow{01}}/\mathbb{Q}(t)\big) & & \pi_1(\mathrm{Spec}\ \mathbb{Q}, \overline{\mathbb{Q}}) & &
\end{array}
$$

§2 Galois Action

2-1 If x : $\mathbb{Q}$-rat'l $\in X$

$\exists s_x$ (OMIT)

2-2 If $x = \overrightarrow{01}$ $\qquad X = \mathbb{P}^1 \setminus \{0, 1, \infty\}/\mathbb{Q}$

$$\xrightarrow{t} \mathbb{C}$$

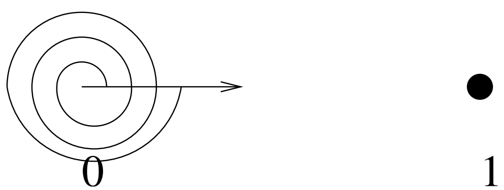
$$s_{\overrightarrow{01}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{G}\big(M_x/\mathbb{Q}(t)\big)$$

$M_{\overrightarrow{01}}$: ① germ around $(0, \epsilon)$

② an. cont.

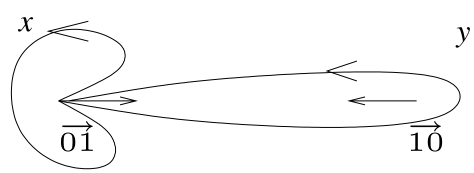
③ alg.

$$
\begin{array}{lcl}
M_{\overrightarrow{01}} \hookrightarrow \overline{\mathbb{Q}}\{\{t\}\} := & \bigcup_{N\in\mathbb{N}} \mathbb{Q}((t^{\frac{1}{N}})) & \overset{\text{act at}}{\underset{\text{coeff}}{\curvearrowleft}} \mathrm{G} \\
\cup & \bigcup & \\
h \in \overline{\mathbb{Q}}((t^{\frac{1}{N}})) & \mathbb{Q}((t)) &
\end{array}
$$



$$
\mathrm{G}_{\mathbb{Q}} \xrightarrow{s_{\overrightarrow{01}}} \mathrm{G}\big(M_{\overrightarrow{01}}/\mathbb{Q}(t)\big) \xrightarrow{\mathrm{Id}} \mathrm{G}\big(\overline{\mathbb{Q}}(t)/\mathbb{Q}(t)\big) \cong \mathrm{G}_{\mathbb{Q}}
$$

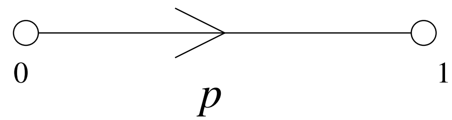
$$\rho_{\overrightarrow{01}} : \mathrm{G}_{\mathbb{Q}} \to Aut(\pi_1(\overline{X}, \overrightarrow{01}))$$
$$\sigma \mapsto s_{\overrightarrow{01}}(\sigma)(\qquad)s_{\overrightarrow{01}}(\sigma)^{-1}$$



(profinite completion) $F_2\hat{} := \pi_1(\overline{X}, \overrightarrow{01}) = \langle x, y\rangle\hat{}$

Theorem: $\rho_{\overrightarrow{01}}(\sigma) : x \mapsto x^{\chi_\sigma}, \quad \chi_\sigma \in \hat{\mathbb{Z}}^\times$
$$y \mapsto f_\sigma^{-1} y^{\chi_\sigma} f_\sigma \ \in [\hat{F}_2, \hat{F}_2]$$

2-3 Groupoid



$$\pi_1(X^{an};\overrightarrow{01},\overrightarrow{10}) \qquad \ni p$$

$$\downarrow \qquad\qquad \mapsto$$

$$Isom_{\overline{\mathbb{Q}}(t)}(M_{\overrightarrow{01}}, M_{\overrightarrow{10}}) \qquad \ni p$$

$$\| \qquad \hookrightarrow$$

$$\pi_1^{alg}(\overline{X};\overrightarrow{01},\overrightarrow{10}) \qquad \overline{\mathbb{Q}}\{\{(1-t)^{\frac{1}{N}}\}\}$$

$$\uparrow$$

$\mathrm{G}_{\mathbb{Q}}$ (called $\rho_{\overrightarrow{01},\overrightarrow{10}}$)

$$\rho_{\overrightarrow{01},\overrightarrow{10}}(\sigma)(p) := s_{\overrightarrow{10}}(\sigma) \circ p \circ s_{\overrightarrow{01}}(\sigma)^{-1} \in \mathrm{Gal}\big(M_{\overrightarrow{01}}/\mathbb{Q}(t)\big)$$

$||$

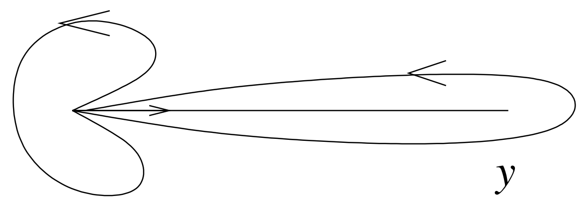
$p \circ (f_\sigma) \in \pi_1(\overline{X}, \overrightarrow{01}) \quad \leftrightarrow \quad$ 1 cocycle

$\overline{\mathbb{Q}}(1-t) \quad \overline{\mathbb{Q}}(t)$

$$x \in \mathrm{Gal}\big(M_{\overrightarrow{01}}/\overline{\mathbb{Q}}(t)\big) \ni h$$

$$h = \sum_{i=-n}^{\infty} a_i t^{\frac{i}{N}} \overset{x}{\longmapsto} \sum a_i \zeta_N^i t^{\frac{i}{N}}$$

$$t^{\frac{i}{N}} \overset{x}{\longmapsto} \zeta_N t^{\frac{i}{N}} = \exp(\tfrac{2\pi i}{N})$$



$$\rho_{\overrightarrow{01}}(\sigma)(x) := \sigma x \sigma^{-1} \quad (\sigma = s_{\overrightarrow{01}}(\sigma))$$

$h \in M_{\overrightarrow{01}}$

$$\begin{array}{ccc} h \xmapsto{\sigma^{-1}} \sum_i \sigma^{-1}(a_i) t^{\frac{i}{N}} & & \\ \big\downarrow x & & \\ \sum \sigma^{-1}(a_i)\zeta_N^i t^{\frac{i}{N}} \xmapsto{\sigma} \sum a_i \sigma(\zeta_N^i) t^{\frac{i}{N}} & & (\sigma(\zeta_N^i) = \zeta_N^{\chi(\sigma) i}) \end{array}$$

$G_{\mathbb{Q}} \ni \sigma \qquad a_{\sigma,N} \in (\mathbb{Z}/N)^\times$

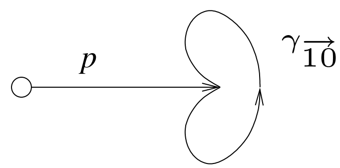
$\sigma(\zeta_N) = \zeta_N$

$$(a_{\sigma,N}) \in \varprojlim_{N} (\mathbb{Z}/N)^{\times}$$

$$||$$

$$\chi(\sigma) \in \hat{\mathbb{Z}}^{\times}$$

$$\rho_{\overrightarrow{01}}(\sigma)(x) := x^{\chi(\sigma)}$$



$$2y = p^{-1} \circ \gamma_{\overrightarrow{10}} \circ p$$

$$\Rightarrow \quad \rho_{\overrightarrow{01}}(\sigma)(y) = \overbrace{\underset{s_{\overrightarrow{01}}}{\underset{\uparrow}{\sigma}}\, p^{-1}\sigma^{-1}}^{f_\sigma^{-1}p^{-1}} \underset{s_{\overrightarrow{10}}}{\underset{\uparrow}{\sigma}}\, \gamma_{\overrightarrow{10}} \underset{s_{\overrightarrow{10}}}{\underset{\uparrow}{\sigma^{-1}}} \overbrace{\sigma p \underset{s_{\overrightarrow{01}}}{\underset{\uparrow}{\sigma^{-1}}}}^{p f_\sigma}$$

$$= f_\sigma^{-1} \underbrace{p^{-1} \gamma_{\overrightarrow{10}}^{\chi(\sigma)} p}_{y^{\chi(\sigma)}} f_\sigma$$

$\rho_{\overrightarrow{01}} : \mathrm{G}_{\mathbb{Q}} \hookrightarrow Aut(\hat{F}_2)$ (Belyĭ)

§3. $\mathrm{G}_{\mathbb{Q}} \curvearrowright \hat{F}_2/\underline{\underline{N}}$ a quotient

$f_\sigma \bmod \underline{\underline{N}} \in \hat{F}_2$

3-1 $\underline{\underline{N}} := [\hat{F}_2, \hat{F}_2] := \hat{F}_2'$

<u>Ex. 1</u> $\overline{f}_\sigma \in \hat{F}_2^{ab}$

$||$

1

$$
\begin{array}{ccc}
\hat{F}_2 & \overset{\sim}{\longrightarrow} & \mathrm{Gal}\big(M_{\overrightarrow{01}}/\overline{\mathbb{Q}}(t)\big) \\
\downarrow & & \downarrow \\
\hat{F}_2/\underline{\underline{N}} & \overset{\sim}{\longrightarrow} & \mathrm{Gal}\big(M_{\overrightarrow{01}}^{\underline{\underline{N}}}/\overline{\mathbb{Q}}(t)\big) \\
 & & \Vert
\end{array}
$$

If $\underline{\underline{N}} = \hat{F}_2'$ then $\overline{\mathbb{Q}}(t^{\frac{1}{N}}, (1-t)^{\frac{1}{N}} | N \in \mathbb{N})$

$f_\sigma : t^{\frac{1}{N}} \longmapsto t^{\frac{1}{N}}$
$(1-t)^{\frac{1}{N}} \longmapsto (1-t)^{\frac{1}{N}}$

$$f_\sigma = p^{-1}\sigma p \sigma^{-1} \qquad \text{(Recall definition!)}$$

$$\circlearrowright \sigma$$

$$t^{\frac{1}{N}} \xrightarrow{\sigma^{-1}} t^{\frac{1}{N}} \longrightarrow (1-(1-t))^{\frac{1}{N}} = \underbrace{(1 - \tfrac{1}{N}(1-t) + \dots)}_{\text{Taylor expansion}}$$