Galois Representations on π_1 and its Lie alg-tion

1. π_1 as a Gal.

X alg. var/$K \subseteq \mathbb{C}$

X^{an} analytification

$x \in X^{an}$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1^{alg}(\overline{X},x) & \longrightarrow & \pi_1^{alg}(X,x) & \longrightarrow & \pi_1^{alg}(\mathrm{Spec}\ K,x) & \longrightarrow & 1 \\
 & & \| \quad \uparrow & & & & \|\wr & & \\
 & & \pi_1(X^{an},x) & & & & \mathrm{Gal}(\overline{K}/K) & & \\
 & & & & & & \| & & \\
 & & & & & & \mathrm{Aut}(\overline{K}/K) & &
\end{array}$$

1.1 M_x

$\mathcal{M}_x :=$ {germs of meromorphic fct. around x}

$\uparrow$ (analytic sense)

$\bigcup|$

{those can be analytically continued to whole X^{an} as multi-valued function}

$\bigcup|$

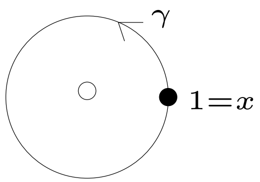
{those algebraic over $\overline{K}(X)$}$=:M_x$

Ex. $X^{an} = \mathbb{C} \setminus \{0\} \xrightarrow{t} \mathbb{C} \qquad K = \mathbb{Q},\ \overline{K}(X) = \overline{\mathbb{Q}}(t)$

$x = 1$

- $e^{\frac{1}{t}} \in \mathcal{M}_x$

- $t^{\frac{1}{N}} = \exp(\frac{1}{N} \log t)$ can be analytically continued



$$z = t^{\frac{1}{N}} \xmapsto{\gamma} \zeta_N t^{\frac{1}{N}}$$

$$\zeta_N = \exp(\frac{2\pi i}{N})$$

- $z^n - t = 0$

- $(t-1)^{\frac{1}{N}}$ cannot be analytically continued

- M_x : the maximal algebraic extension of $\overline{K}(X)$ unramified over x

- $$\begin{aligned} \pi_1(X^{an}, x) &\to \mathrm{Gal}\big(M_x/\overline{K}(X)\big) \\ \gamma &\mapsto \gamma^{an} \end{aligned}$$

- Fact $\forall N \lhd \pi_1(X^{an}, x)$
  M_x^N
  $\pi_1(X^{an}, x)/N \xrightarrow{\sim} \mathrm{Gal}\big(M_x^N/\overline{K}(X)\big)$

  Grothendieck's Riemann Existence Thm.

$$\varprojlim \mathrm{Gal}\big(M_x^N/\overline{K}(X)\big) \simeq \mathrm{Gal}\big(M_x/\overline{K}(X)\big)$$

$$\|\wr$$

$$\varprojlim_N \pi_1(X^{an},x)/N =: \underbrace{\pi_1(X^{an},x)\hat{}}_{\text{profinite completion}}$$

$$=: \pi_1^{alg}(\overline{X},x)$$

$$\overline{X} := X \underset{K}{\otimes} \overline{K}$$

- $M_x \supseteq \overline{K}(X) \supseteq K(X)$

  $\nwarrow \quad \nearrow$

  Galois

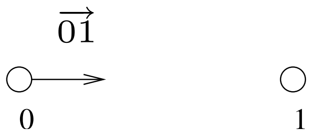
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{G}\big(M_x/\overline{K}(X)\big) & \longrightarrow & \mathrm{G}\big(M_x/K(X)\big) & \longrightarrow & \mathrm{G}\big(\overline{K}(X)/K(X)\big) & \longrightarrow & 1 \\
 & & \Big\| \wr & & \Big\| \wr & & \Big\| \wr & & \\
 & & \pi_1^{alg}(\overline{X}, x) & & \pi_1^{alg}(X, x) & & \mathrm{G}\big(\overline{K}/K\big) & & \\
 & & \Big\| & & & & & & \\
 & & \pi_1(X, x)\hat{} & & & & & &
\end{array}
$$

- $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ $\qquad \mathrm{G}\big(\overline{K}(X)/K(X)\big) = \mathrm{G}\big(\overline{\mathbb{Q}}(t)/\mathbb{Q}(t)\big) \simeq \mathrm{G}\big(\overline{\mathbb{Q}}/\mathbb{Q}\big)$
  $x = 1 \ (t = 1)$
- $M_x = \overline{\mathbb{Q}}(t^{\frac{1}{N}} \in \mathcal{M}_x | N \in \mathbb{N})$ $\qquad \overline{K}(X) = \overline{\mathbb{Q}}(t)$
- to fix $t^{\frac{1}{N}} \in \mathcal{M}_x$ is to fix a branch of $t^{\frac{1}{N}}$ so that $t^{\frac{1}{N}}$ at $1 = x$ is 1 $\qquad$ ~~(ζ_N)~~

$$\mathcal{M}_x \supseteq \overline{\mathbb{Q}}(t^{\frac{1}{N}} | N \in \mathbb{N}) \supseteq \overline{\mathbb{Q}}(t)$$

$$\begin{array}{ccc}
\hat{\mathbb{Z}} & & \\
\| \wr & & \\
\pi_1(X, x)\hat{} & \xrightarrow{\sim} & \mathrm{Gal}\big(M_x/\overline{\mathbb{Q}}(t)\big) \\
& \searrow^{\sim} & \downarrow\!\!\!\downarrow \sim \\
& & \mathrm{Gal}\big(\overline{\mathbb{Q}}(t^{\frac{1}{N}} | N \in \mathbb{N})/\overline{\mathbb{Q}}(t)\big) \\
& & \| \wr \\
& & \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/N\mathbb{Z}
\end{array}$$

<u>1.2</u> $\overrightarrow{01}$-tangential base point $\qquad X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$



- $\mathcal{M}_{\overrightarrow{01}} :=$ { germs of meromorphic functions defined around $(0, \epsilon)$ for some $\epsilon > 0$ }

  $\bigcup|$

  { those that can be analytically continued to X }

  $\bigcup|$

  $M_{\overrightarrow{01}} =$ { those algebraic/$\overline{K}(X) = \overline{\mathbb{Q}}(t)$ }

  $||\wr$ (non canonically)

  M_x

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{G}\big(M_{\overrightarrow{01}}/\overline{K}(X)\big) & \longrightarrow & \mathrm{G}\big(M_{\overrightarrow{01}}/K(X)\big) & \longrightarrow & \mathrm{G}\big(\overline{K}(X)/K(X)\big) & \longrightarrow & 1 \\
 & & \Vert & & \Vert & & \Vert\wr & & \\
 & & \pi_1(\overline{X},\overrightarrow{01}) & & \pi_1^{alg}(X,\overrightarrow{01}) & & \mathrm{G}(\overline{K}/K) & &
\end{array}$$