Slope:

$$Z(U_{f_\lambda}, T) \sim \prod (1-\alpha_i(\lambda)T)$$

$$\alpha_i(\lambda) \in \overline{Q}.$$

1) $\alpha_i(\lambda) \hookrightarrow \mathbb{C}$.

$$\alpha_i(\lambda) = |\alpha_i(\lambda)| \cdot e^{i\theta(\lambda)}$$

$$= q^{w_i(\lambda)} \cdot e^{i\theta(\lambda)}.$$

$w_i(\lambda)$ is constant. $\hookrightarrow$ Hodge numbers $h^{p,q}$.

2) $\alpha_i(\lambda) \hookrightarrow \mathbb{C}_\ell, \quad \ell \neq p.$

$$|\alpha_i(\lambda)|_\ell = 1.$$

3). $\alpha_\phi(\lambda) \hookrightarrow \mathcal{O}\, \mathbb{C}_p$.

$$\alpha_\phi(\lambda) = q^{-s_\phi(\lambda)} \cdot u_\phi(\lambda). \qquad |u(\lambda)|_p = 1$$

slope $s(\lambda) = \mathrm{ord}_q\, \alpha(\lambda)$. $\in \mathbb{Q}$.

depends on λ, in mild way.

(only finite # of possibility)

$\Rightarrow$ Newton polygon $NP(f)$ is a finer invariant than $HP(\Delta)$, but cruder than zeta.

$$GNP(\Delta, p) \stackrel{?}{=} HP(\Delta).$$

<u>Conj.</u> $\exists$ a positive integer ~~...~~ $u(\Delta)$

s.t. $\{p > d'(\Delta) \mid p \text{ ordinary}\}$

$=$ disjoint union of some congruence classes mod $u(\Delta)$.

(8). ordinary primes for reflexive Δ.

Def. $\Delta \subseteq \mathbb{R}^n$, n-dim integ. convex

$$\Delta^* = \{(x_1, \ldots x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i y_i \geq -1, \ \forall y \in \Delta\}.$$

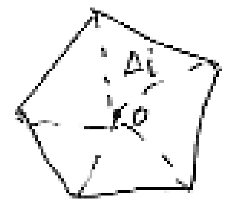
$\Rightarrow \Delta^*$ is convex, but not integral.

Def. Δ is reflexive if Δ^* is also integral.

$(\Delta^*)^* = \Delta$.

Def. Δ is Fano if reflexive.

Δ_i is a simplex
with $d(\Delta_i) = 1$.



Thm B. Let Δ be reflexive.

1) If $n \leq 4$. $\Rightarrow$ Δ is ordinary $\forall\ p > d(\Delta)$.

2) If Δ is Fano, $\Rightarrow$ Δ is ordinary for all $p > d(\Delta)$ $\cancel{=}$ $\cancel{n+1}$.

$= \#\{$of Codim 1 faces$\}$

Thm (star decomp). Δ reflexive.

$$\Delta = \bigcup_{i=1}^{k} \Delta_i$$

If p is ordinary $\forall \Delta_i$.

$\Rightarrow$ p is ordinary $\forall \Delta$.

? If Reflexive Δ has dim $= n \geq 5$,

Is Δ ordin for $p > d(\Delta)$?

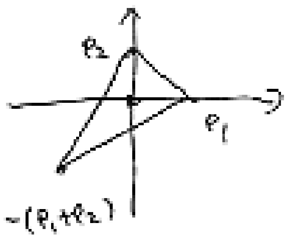
19. Basic example.

$$f(\lambda, X) = X_1 + X_2 + \cdots + X_n + \frac{1}{X_1 \cdots X_n} - \lambda.$$

$$\Delta = \Delta(f) = \langle e_1, e_2, \cdots, e_n, -(e_1 + \cdots + e_n) \rangle.$$

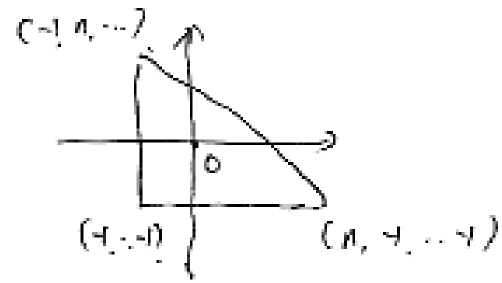
$$\Delta^* = \langle (n, -1, \cdots, -1), \cdots, (-1, -1, \cdots, n), (-1, -1, \cdots, -1) \rangle$$

Δ.



reflexive.

Δ^*.

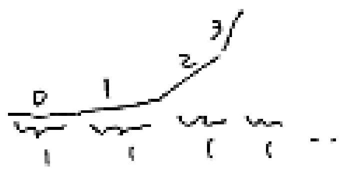


Δ is Fano. Δ^* is NOT Fano $(n > 1)$

$f(\lambda, x)$ is Δ-regular, $\Longleftrightarrow$ $\lambda \neq (n+1)\alpha$, $\alpha^{n+1} = 1$.

$d(\Delta) = n+1$. $h(0) = h(1) = \cdots = h(n) = 1$.

$Hp(\Delta)$.



$\sum h(i) = n+1$.

Thm. ~~Let $p \nmid (n+1)$~~. $f(\lambda, x)$ is Δ-regular $/ \overline{\mathbb{F}}_q$. $\Rightarrow$

1) $L(\chi, f, T)^{(-1)^n} = \prod_{i=0}^{n} (1 - \alpha_i(\lambda) T)$.

2) $\alpha_0(\lambda) = 1$. $|\alpha_i(\lambda)| = q^{\frac{n+1}{2}}$ $(1 \le i \le n)$.

3) Generically ordinary $\forall\, p \nmid (n+1)$.

For all but finitely many λ, $\Rightarrow$

$\mathrm{ord}_q(\alpha_i(\lambda)) = i$.

Question: 1). How $\check{Z}(\upsilon_{f_\lambda}, T)$ varies with λ?

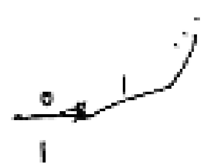
2) How $\alpha_i(\lambda)$ varies with λ?

20). Zeta functions.

$$f \in \mathbb{F}_q[X_1^{\pm 1}, \ldots, X_n^{\pm 1}], \quad \Delta\text{-regular} / \mathbb{F}_q.$$

$$L(x \circ f, T)^{(-1)^n} \Big|_{T=1} = 0.$$

$$\Rightarrow \frac{L(x \circ f, T)^{(-1)^n}}{1-T} = P(f, qT),$$



$$P(f, T) \in 1 + T\mathbb{Z}[T]$$

of deg $d(\Delta) - 1$.

$$Z(U_f, qT) = \prod_{i=0}^{n} (1-q^iT)^{(-1)^{n-i-1}\binom{n}{i}} L(x_0f, T)$$

$$= \prod_{i=1}^{n} \cdots \left(\frac{L(x_0f,T)^{(-1)^n}}{1-T}\right)^{(-1)^n}$$

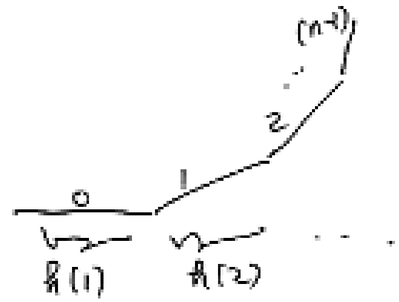
$$\Rightarrow Z(U_f, T) = \prod_{i=0}^{n-1} (1-q^iT)^{(-1)^{n-i}\binom{n}{i+1}} P(f,T)^{(-1)^n}.$$

$$P(f, T) = \prod_{i=0}^{d(\Delta)-2} (1 - \beta_i T)$$

$$\Rightarrow \quad \# U_f(\mathbb{F}_{q^k}) = \frac{(q^k - 1)^n + (-1)^{n+1}}{q^k} + (-1)^{n+1} (\beta_0^k + \beta_1^k + \cdots + \beta_{d(\Delta)-2}^k).$$

$$(k = 1, 2, 3, \ldots)$$

Def. The primitive Hodge polygon $PHP(\Delta)$ is



$$NP(P(f,T)) \geq PHP(\Delta).$$