- Introduce A - a hypergeometric system
- Take $A \subset M \approx \mathbb{Z}^d$ a finite subset.
- $\langle A, u \rangle = 1$ for all $u \in N$, the dual lattice. This is the regular case.
- Begin with $f(x) = \sum_{m \in A} a_m X^m$, and consider the integral

$$I(\beta, \{a_m\}) = \int_C x^{-\beta} \exp(f(x)) \frac{dx}{x},$$

for some $\beta \in M_{\mathbb{C}}$, over a cycle $C \in \mathbb{T}^d(\mathbb{C})$.

- What are some differential equations satisfied by $I(\beta, A)$?

- Let $A = \{v_1, \dots, v_n\}$ and the map $\mathbb{Z}^n \to M$ which takes the i-th standard basis element e_i to v_i.

- This gives the exact sequence

$$R(A) \to \mathbb{Z}^n \to M,$$

where $R(A) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n | \sum \lambda_i v_i = 0\}$, and has rank $n - d$. We call this the lattice of relations.

- For $(\lambda_i) \in R(A)$, we consider the operators $\prod\limits_{\lambda_i < 0} (\frac{\partial}{\partial a_i})^{\lambda_i}$ and $\prod\limits_{\lambda_i > 0} (\frac{\partial}{\partial a_i})^{\lambda_i}$, and note that, by regularity, applying either to $I(\beta, a)$ gives the same result.

- Thus the difference operator $\Box_\lambda$ is 0 on $I(\beta, a)$.

- Take $\beta \in M_{\mathbb{C}}$ or $\beta \in M_{\mathbb{Q}}$, and choose a basis for the dual lattice N, $\{u_1, \ldots, u_d\}$.

- Define for $1 \leq k \leq d$,

$$D_k^{\beta} = \sum_{i=1}^{n} \langle v_i, u_k \rangle u_i \frac{\partial}{\partial a_i} - \langle \beta, u_k \rangle$$

.

- Then $D_k^{\beta} I(\beta, a) = 0$.

- We collect now the restrictions that $\Box_{\lambda} \phi(a) = 0$, and $D_i^{\beta} \phi(a) = 0$, for $1 \leq i \leq d$. This is what we call an A-hypergeometric system.

Simplest Example of a Regular Hypergeometric System:

- Let $A = \{0, 1, \ldots, n\} \in \mathbb{Z}$ to be A, which become the coefficients in
$$f(x) = \sum_{i=0}^{n} a_i X^i.$$
- Here again $\tilde{M} = \mathbb{Z} \oplus M$, giving our usual triangle.
- A is the set of vectors.
- We have the relation $v_i + v_j = v_k + v_\ell \iff i + j = k + \ell$, and that if $i + j = k + \ell$, that
$$\left(\frac{\partial}{\partial a_i} \frac{\partial}{\partial a_j} - \frac{\partial}{\partial a_k} \frac{\partial}{\partial a_\ell} \right) \phi(a) = 0$$
- Moreover, if $\beta = (\beta_1, \beta_2) \in \tilde{M} = \mathbb{Z}^2$, then $\left(\sum_{i=0}^n a_i \frac{\partial}{\partial a_i}\right) \phi = \beta_1 \phi$ and $\left(\sum_{i=0}^n i a_i \frac{\partial}{\partial a_i}\right) \phi = \beta_2 \phi$.

- Consider the hypersurface $\overline{Z}_f \subset \mathbb{T}^d$ defined by $f = 0$.
- We take a cycle $\gamma \in H_{d-1}(Z_f, \mathbb{Z})$ and take $\omega \in H^{d-1}(Z_f)$.
- Then $\int_\gamma \omega$ satisfies a hypergeometric system as a function of the coefficients.
- Restriction gives the map

$$\Omega^{d-1} Z_f \leftarrow \Omega^d(\mathbb{T}^2 \backslash Z_f)$$

on differential forms, defining the form $\frac{x^n}{(f)^k}\frac{dx}{x}$.

Back to the Regular example:

- $f(x) = \sum a_i X^i$, so Z_f is the set of complex roots of $f(x)$.
- A theorem from Mayer states that the μ-th powers of the roots, ρ_1^μ, $\rho_2^\mu, \ldots, \rho_n^\mu$ are a solution to the A-hypergeometric system with $\beta = (0, -\mu)$.

How do we solve polynomial equations?

- For $n = 2$, we have $R(A) = \langle(1, -2, 1)\rangle$, and the relations generated by $\frac{\partial}{\partial a_0}\frac{\partial}{\partial a_2} - \left(\frac{\partial}{\partial a_i}\right)^2$.

- For the polynomial $a_0 + a_1 x + a_2 x^2$, we recover the discriminant $a_1^2 - 4a_0a_2$. If $a_1^2 >> 4a_0a_2$, we can write the discriminant as $a_1\sqrt{1 - \frac{4a_0a_2}{a_1^2}}$, so letting $z = \frac{a_0a_2}{a_1^2}$, we just need to expand $\sqrt{1-4z}$ in power series.

- Else, if $a_0a_2 >> a_1^2$, we expand $\sqrt{1 - \frac{w}{4}}$ for $w = \frac{a_1^2}{a_0a_2}$.

- In general, we just need to isolate the leading monomial. To do this, we use the Newton polytope.

- Considering the secondary polytope, the power series expansion comes from the expansion of some cone, as in the second lecture.

- For $n = 2$, the secondary polytope was the line segment with vertices labeled a_1^2 and $-2^2 a_0 a_2$.

- For $n = 3$, we get as before a trapezoid with vertices labeled by the monomials appearing in the discriminant.

- In general, $\mathrm{Sec}(A)$ is topologically an $(n-1)$-dimensional cube with 2^{n-1} points (Stumfel).

- In the regular case, for a generic β, consider the polytope Δ, the convex polytope of A, so Δ has dimension $d-1$.

- Take $d=3$. For each triangulation, we have a power series

$$\phi_\alpha(a) = \sum_{\ell \in R(A)} \prod_{i=1}^{n} \Gamma(\alpha_i + \lambda_i + 1)^{-1} a^{\lambda+\alpha}.$$

- We can choose $\alpha \in \mathbb{C}^n$, so that we get $\beta \in M_{\mathbb{C}}$ in the analytification, and for non-resonant β, we find $\mathrm{Vol}(\Delta)$ independent solutions.

- The size of the secondary polytope is $|A| - d$.

- We now have solutions via power series and via integrals. What's the link?