- Start with $A \subset M$, $f(x) = \sum_{m \in A} a_m x^m$.
- $\Delta(f) = \mathrm{Conv}(A)$.
- Non-degeneracy condition (NDC): $f_\Theta(x) = x_i \frac{\partial f}{\partial x} \Theta_i = 0$ should have no solutions.
- If $f_\Theta = f$, $\Theta = \Delta$, we need $Z_f \subset \mathbb{T}^d$ non-singular.

Example:

- $A = [0, 1, \ldots, n]$, $\Delta = [0, n]$. $f(x) = \sum_{i=0}^{n} a_i x^i$.
- NDC: $a_0 a_n D(f) \neq 0$, where $D(f)$ is the discriminant.

More generally, we consider the determinant $\det(C)$
of the Koszul complex:
We take the $d + 1$ elements $F_0, F_1, \ldots, F_d \in S_\Delta^1 \subset S_\Delta$,
a resolution of $S_f = S_\Delta / \langle F_0, \ldots, F_d \rangle S_\Delta$.

- Consider $f = \sum a_m x^m$, $R = \mathbb{Z}[a_1, \ldots, a_n]$, where $n = |A|$.
- The graded pieces form a Koszul complex.
- $S_f \leftarrow S_\Delta \leftarrow S_\Delta(-1) \leftarrow \cdots$, of the Artin ring S_f.
- Similarly, we construct $S_f^k \leftarrow S_\Delta^k \leftarrow \Lambda^1 \tilde{N} \otimes S_\Delta^{k-1} \leftarrow \cdots$
- Take the determinant $\det C_\bullet^k(f)$.
- Theorem: $\det(C_\bullet^k)$ is independent of k for $k >> 0$, and $E_A(f) \equiv \det C_\bullet^k(f) \in R$ is called the principal A-determinant of f.

- For the example $A = [0, \ldots, n]$, we get that $E_a(f) = a_0 a_n D(f)$.
- As another example, let Δ be a simplex, A be its vertices, and $f = \sum a_m x^m$.
- Then $E_A(t) = \pm(\mathrm{Vol}(\Delta))^{\mathrm{Vol}(\Delta)} \cdot (a_0 a_1 \cdots a_d)^{\mathrm{Vol}(\Delta)}$.
- $\mathrm{Vol}(\Delta)$ is $d!$ times the normal volume.
- Example: The unit square. $E_A(f) = a_0 a_1 a_2 a_3 (a_1 a_3 - a_0 a_2)$.

- $E_A(f)$ is homogeneous of degree $(d+1)\mathrm{Vol}(\Delta)$.
- Example: $d = 1, \Delta = [0, n]$.
  Then $\deg E_A(f) = 2n$ and $\deg \Delta(f) = 2n - 2$.
- There are d more homogeneity conditions, coming from the action of $\mathbb{T}^d$.
- The classical formula: $D(f) = a_n^{2n-2} \prod_{i<j} (\rho_i - \rho_j)^2$, where $\rho_1, \ldots, \rho_n$ are the roots of f.
- Act by the torus: $\Pi^1 \times f$ by
  $(\lambda \cdot f) \to a_0 + \lambda a_1 x + \cdots + \lambda^n a_n x^n$.

- Define $wt(a_i) = i$. Then $D(f)$ is quasi-homogeneous of degree $n^2 - n$:
(A function is quasi-homogeneous if it is homogeneous with respect to weights.)
- $D(f) \to E_A(f) = a_0 a_n D(f)$. (More elegant).
- More generally, $E_A(f) =$ "$\prod_{\Theta \subset \Delta} D(f_\Theta)$."

Secondary Polytope

- The secondary polytope: Newton polytope of $E_A(f)$.
- Let $n = |A|$, a_i be monomials, $E_A(f) = \sum c_k a^k$.
- Lattice points generated by monomials: $L \subset \mathbb{Z}^n$, a hyperplane of codimension $d + 1$.
- $\mathrm{Sec}(A) =$ the convex hull of all μ with nonzero coefficients.

- Gelfand-Kapranov-Zelevinski theorem describes all vertices of Sec(A). (Statement deffered)
- Take $d = 1$, and take the Newton diagram of a 1-dimensional deformation of f: Take $m_i \in \mathbb{Z}_{>0}$, and $f_t(x) = t^{m_0} + \cdots + t^{m_n} x^n$.
- Then $D(f_t) = ct^{\epsilon}(1 + o(t))$.
- Compute $D(t^{m_0}, \ldots, t^{m_n})$ via $(a_i \to t^{m_i})$.
- This discriminant equals $\sum c_k t^{\langle k, m_i \rangle}$.
- Or: $D(f) = a_n^{n-2} \prod_{i<j} (\rho_i - \rho_j)$.
  Write $\rho_i = c_i t^{\epsilon_i}(1 + o(t))$ (if we know the Newton diagram).

The Newton Diagram:

- $f_t(x) = F(x, t)$.
- This convex hull is the Newton diagram.
- The diagram uniquely determines the expression for the roots.
- So vertices of the secondary polytope correspond to types of Newton diagrams.
- Def: Two Newton diagrams are equivalent if their projections give the same partition of the interval.

- An equivalence class of diagrams is given by a partition of $I = [0, 1, \dots, n]$.
- There are 2^{n-1} vertices of the cube of dimension $n-1$, so there are 2^{n-1} types.
- GKZ Theorem: Generalization of this correspondence, where the polytope is multidimensional.
- Write $f_t(x) = \sum_{m \in A} t^{\phi(m)} x^m$.
- These $\phi(m)$ generalize the a_m.

Generalizing to GKZ:

- We have a convex $\phi : A \to \mathbb{Z}_{>0}$ instead of the a_m.
- Triangulations generalize partitions of intervals.
- GKZ Theorem: Vertices $\iff$ admissible triangulations.
- Further, monomials in $E_A(f) \iff$ vertices of $\mathrm{Sec}(A)$.
- We are interested in $\prod_{\dim T = d} E_T(a)$, the product of principal determinants of each face.
- From before $E_T(a) = (\mathrm{Vol}(T))^{\mathrm{Vol}(T)}(a_0 \cdots a_k)^{\mathrm{Vol}(T)}$.
- The coefficients are $\prod_{\dim T = d} \mathrm{Vol}(T)^{\mathrm{Vol}(T)}$.

- We need to check triangulations don't come from projections of boundaries.
- Action: $\mathbb{T}^d \times E_A(f) = \chi B_H(t)$.
- Here χ is the character of a torus, given by β, where β is the barycenter of the polytope Δ times $(d_1)\mathrm{Vol}(\Delta)$.