Definitions and Notation:

- $M \approx \mathbb{Z}^d$ is a lattice, $N = \mathrm{Hom}(M, \mathbb{Z})$,
- A pairing $\langle *, * \rangle : M \times N \to \mathbb{Z}$.
- A finite subset $A = \{v_1, \ldots, v_m\} \subset M$.
- The vector space $M_{\mathbb{R}} = M \otimes \mathbb{R}$.
- $\Delta = \mathrm{Conv}(A) \subset M_{\mathbb{R}}$ is the convex hull of A in $M_{\mathbb{R}}$.

- Special/Regular Case: $\dim \Delta = d - 1$ such that $\exists u \in N$ with $\langle u, v_i \rangle = 1$ for all $i = 1, 2, \ldots, n$.

- General case with dim $= d$ can be reduced to a regular case with dim $= d + 1$:

$$M \quad \rightarrow \quad \tilde{M} = \mathbb{Z} \oplus M \tag{1}$$

$$\tilde{N} = \mathbb{Z} \oplus N \tag{2}$$

$$\tag{3}$$

  And a new pairing: $\langle \tilde{x}, \tilde{y} \rangle \rightarrow \mathbb{Z}$.

- Take a polytope $\Delta \subset M_{\mathbb{R}}$ to $\tilde{\Delta} = \mathrm{Conv}((1, \Delta))$.

- Example: Take $\Delta = [0, n] \cap \mathbb{Z}$ into the line $y = 1$ in $\mathbb{R}^2$.
- Reverse process: General case of dimension d to a regular of dimension $d - 1$ by restriction.
- Main Idea: A Laurent polynomial $f(x) = \sum_{m \in A} a_m x^m$ with $a_m \in K$, so $f \in k[M] \approx k[x_1^{\pm 1}, \dots, x_n^{\pm n}]$.
- Could take $k = \mathbb{R}, \mathbb{C}, \mathbb{F}_p$, etc.

- Consider affine space $\mathbb{A}_k^{|A|} \subset \mathbb{A}_k^{\#(\Delta \cap M)}$.
- $\Delta = \mathrm{Conv(A)}$, $A \subset (\Delta \cap M)$.
- A contains all the vertices of Δ.

Non-Degeneracy:

- Open condition: For any face $\theta \subset \Delta$, define $f_\theta(x) = \sum_{m \in \theta \cap A} a_m X^m$.
- Definition: f is Δ-nondegenerate if for any face,

$$f_\theta = x_i \frac{d}{dx_i} f_\theta = 0 \tag{4}$$

  has no solutions for all i in the torus over $\overline{k}$.

- Example: $A = \{0, 1, \ldots, n\} \in \mathbb{Z}$, $f(x) = \sum a_i x^i$. $\Delta = [0, n]$, Faces: $[0, n], \{0\}, \{n\}$, with $a_0 a_n \neq 0$, and $f(x) = f'(x) = 0$ has no solutions.

Toric Geometry: $\mathbb{T}^d \subset \mathbb{P}_\Delta$.

- $A = \{v_1, \ldots, v_m\}$. Assume $\{v_i - v_1\}$ generate M.
- Embed $\mathbb{T}^d \subset \mathbb{P}^{|A|-1}$ via $x \to (x^{v_1} : \cdots : x^{v_n})$.
- $\overline{\mathbb{T}^d} = \mathbb{P}_\Delta$, $\mathbb{P}_\Delta = \mathrm{Proj} S_\Delta$.
- A cone $C_\Delta \subset \tilde{M}_\mathbb{R}$, generated by $(1, v_1), \ldots (1, v_n)$.
- The semigroup ring of $C_\Delta \cap \tilde{M}$ under addition:
  $K[C_\Delta \cap \tilde{M}] = S_\Delta$.
- For our previous example: $\{(1,0), \ldots, (1,n)\}$.

Geometric Interpretation

- Take $Z_f \subset \mathbb{T}^d$ defined by $f = 0$,
  compactification gives $\overline{Z_f}$ and P_Δ.
- $P_\Delta = \bigcup_{\theta \subset \Delta} \mathbb{T}_\theta^{\dim \theta}$.
- Define $Z_{f,\theta} = \mathbb{T}_\theta^{\dim \theta} \cap \overline{Z_f} \subset \mathbb{T}_\theta^{\dim \theta}$ defined by f_θ.
  Hypersurfaces are smooth of codimension 1.

Example

- Take $\Delta = \text{Conv}(\{(0,0),(0,n),(n,0)\})$.
- $\mathbb{P}_\Delta \approx \mathbb{P}^2 \longrightarrow \mathbb{P}^{n(n+1)/2-1}$.
- $Z_f \to \overline{Z_f}$ curve of degree n.
- $Z_i = 0$, $i = 0, 1, 2$, the line is $\mathbb{P}^2$.
- The curve $\overline{Z_f} \cap \ell_i$ is transversal.

Commutative Algebra

- $A \to F_0, \ldots, F_d \in S^1_\Delta \subset S_\Delta \subset K[x_i^{\pm 1}]$, $i = 1, \ldots, d$.
- $F_0 = x_0 f(x)$, $F_i = x_i \frac{\partial}{\partial x_i} F_0$.
- Theorem: f is Δ-nondegenerate iff $F_0, \ldots, F_d$ is a regular sequence in S_Δ.
- i.e. multiplication by F_i is injective for all i in $S_\Delta / \langle F_0, \ldots, F_{i-1} \rangle S_\Delta$.
- S_Δ is a Cohen-Macaulay ring of Krull dimension $d + 1$.
- For the triangle example, $F_0 = x f(x)$, $F_1 = x_0 x \frac{df}{dx}$.

- Take N, $\tilde{N} = \mathbb{Z} \oplus N$.
- Embed $M \to \tilde{M}$ by $x^M \to \frac{dx^m}{x^m}$.
- Identify $M \otimes K = \Omega^1(\mathbb{T}^d)_{(1)}$.
- $\mathrm{Lie}(\mathbb{T}^d) \approx N \otimes K$, $\mathrm{Lie}(\mathbb{T}^{d+1}) = \tilde{N}$.
- $S_\Delta(*, -) \otimes \Lambda^* \tilde{N}$.

Koszul Sequence

- We have a sequence
  $\cdots \to S_\Delta(-2) \to S_\Delta(-1) \to S_\Delta$, corresponding to
  $\cdots \to S_\Delta(-2) \otimes \Lambda^2 \tilde{N} \to S_\Delta(-1) \otimes \Lambda^1 \tilde{N} \to S_\Delta \otimes \Lambda^0 \tilde{N}$
- Take a basis $u_1, \ldots, u_{d+1}$ of $\tilde{N}$.
- $\Lambda^{s+1}\tilde{N}$: $d(u_{i_0} \wedge \cdots \wedge u_{i_s})$ is given by
  $\sum_{j=0}^{s} (-1)^j \partial_{u_{i_j}} F_0 \wedge u_{i_1} \wedge \cdots \wedge \hat{u_{i_s}} \wedge \cdots \wedge u_{i_r}$.
- Differential $\partial_u F_0 = \sum_{m \in A} \langle (1,m), u \rangle_{a_m} x^{(1,m)}$.
- Theorem: Koszul complex is acyclic except for degree $i = 0$.
- $H_i(C_0(H)) = S_\Delta / \langle F_0, \ldots, F_{i-1} \rangle S_\Delta = S_f$.

Determinant of a Complex

- R is an integral domain, $K = \mathrm{Frac}(R)$.
- Take a complex of locally free R-modules
  $\cdots \to C_k \to C_{k-1} \to \cdots \to C_1 \to C_0$
  which is exact after tensoring with K.
- $\det(C_\bullet) \subset k$ is an R-module.
- For length 1: $0 \to C_1 \to C_0 \to 0$.
- $C_1 \otimes k$ and $C_0 \otimes k$ are vector spaces of rank 1.
- $\det(C) = \det(\text{change of base matrix})$.