Lecture 5

2-parameter example
"octic"

$\mathbb{P}^{1,1,2,2,2}$ $\quad P(x_1,\dots,x_5)$ = wtd degree 8.

Introduce x_6 (homog. coord used to resolve sings)

Introduce X_0 (for GLSM)

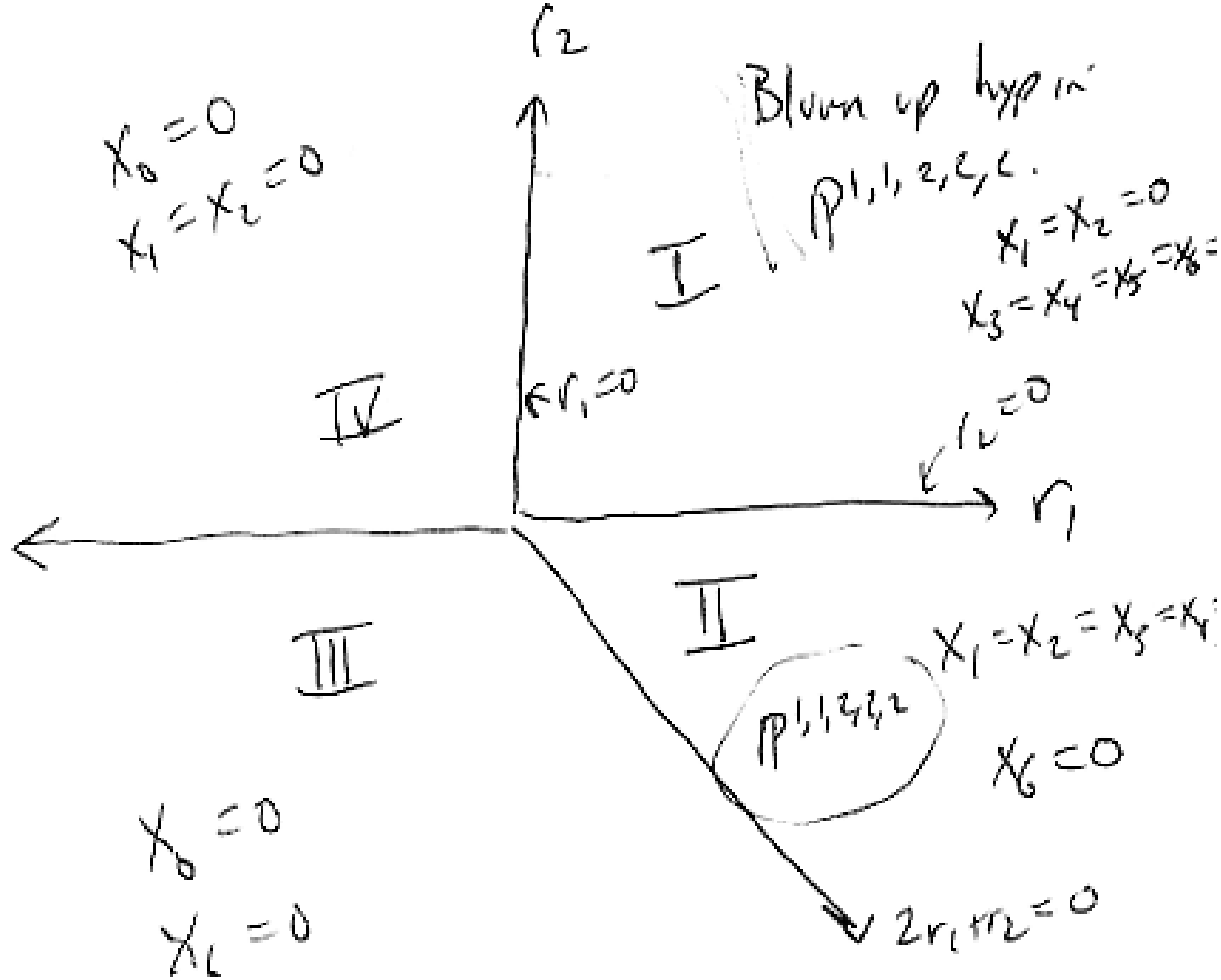
$$W(X_0, X_1, \ldots, X_6) = X_0 P(X_1, \ldots, X_6)$$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$u(1)$	-4	0	0	1	1	1	1
$u(1)$	0	1	1	0	0	0	-2

$$[\,-8 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2 \quad 0\,]$$

$$\mu^{-1}(r)/G \cong \mathbb{C}^n - Z(r)/G_{\mathbb{C}}.$$

($\mathfrak{g}^*$ has coords r_1, r_2)

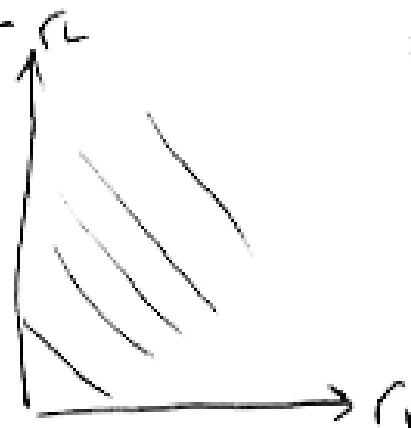

r2
Blown up hyp in
I
II
III
IV
x0 = 0
x1 = x2 = 0
r1 = 0
r2 = 0
r1
x6 = 0
2r1+r2 = 0

(can eliminate X_6, $\mathbb{C}^5/\Gamma$)

Instanton sums:

$\mathbb{CP}^1 \to X$

$f : \mathbb{C}^2 \to \mathbb{C}^n$



$$\frac{\partial W}{\partial x_i} = 0 \quad \Rightarrow \quad \begin{array}{l} x_0 = 0 \\ p(x_1, \dots, x_6) = 0. \end{array}$$

$X_0 = 0 \Leftarrow F_0(s,t)$ has degree $d_0 < 0$

$P(f_1(s,t), \ldots, f_6(s,t))$ = poly in s,t of degree d_0.

make it vanish at $-d_0+1$ points

$(+K)^{-d_0+1}$

$-\sum \delta_i = K$.

$x_0 = 0$ can be imposed on the

moduli spaces of instantons.

$\mathbb{C}^2 \longrightarrow \mathbb{C}^{n-1}$ setting $x_0 = 0$.

toric analysis: $\mu_0^{-1}(r)/G$ is compact

if we're in region I

$$\langle\langle D_1 D_2 D_3 \rangle\rangle = \sum_{q^{\vec{n}}} \langle D_1 \cdot D_2 \cdot D_3 \cdot (\xi)^{-d_j-1}$$

makes sense because instanton moduli spaces are compact

if $d_j < 0$, need $x_{\vec{n}} = \prod_{d_j<0} \xi_j^{-d_j-1}$

Instanton sum calculations in
regions II, III, IV ...

Method: proposed by DRM + Ronen Plesser

(not a complete derivation).

Observation: form of

$$(K)^{-d_0+1} = (K)^{-d_0-1+2}$$

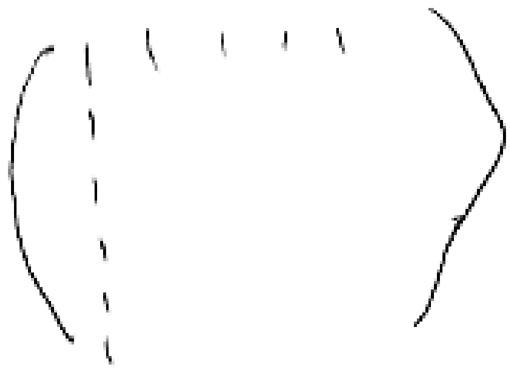
and of

$$\prod_{d_j<0} \xi_j^{-d_j-1} \quad d_1, d_2, \ldots, d_n$$

are very similar

$$\boxed{\xi_0 = -\sum_{j>0} \xi_j}$$

$P = ST$



$$\Rightarrow \sum_{j \geq 0} \xi_j = 0$$

$$\left(\prod_{\substack{d_j<0\\ j\geq 0}} \xi_j^{-d_j-1}\right)\xi_0^2$$

$$\sum \left\langle D_1\cdot D_2\cdot D_3\cdot \left(\prod_{\substack{d_j<0\\ j\geq 0}} \xi_j^{-d_j-1}\right)\xi_0^2 \right\rangle_{\vec{n}} q^{\vec{n}}$$

Coho. of compact toric variety:

- algebra with an extra mapping

$$\langle \ \rangle : H^*(X) \to \mathbb{C} .$$

$$\gamma \longmapsto \gamma|_{[X]} .$$

"Frobenius algebra"

$H^2(X)$ generates $H^{ev}(X)$

$$\langle D_1 \cdot D_2 \cdot D_3 \cdot \cdots \cdot D_k \rangle\big|_{[\mathcal{M}_{\vec{n}}]} \in \mathbb{Q}.$$

Formulate instanton calculation, using:

generators $\xi_1, \ldots, \xi_n$ of the algebra

relations: linear relations from G.

(including $\xi_0 = -\sum \xi_j$)

nonlinear relations

$$\xi_{i_1} \cdots \xi_{i_g} = 0$$

if $(x_{i_1} = \cdots x_{i_g} = 0)$ is a component $\mathcal{Z}$.

This does <u>not</u> determine a Frobenius algebra.

New ring <u>structure</u>: $\overline{D_1 \cdot D_2} = D_1 \cdot D_2 \cdot \xi_0 \in$ algebra

algebra / Ker (mult. by ξ_0).

Leads to formal calculations for
instanton sums in every region;
agree with original in geometric regions
(e.g. region I).

Quintic case:

$$\frac{5}{1+5^5 q} = \sum (-5)^{-5} q^n$$

$$|q| = e^{-2\pi r} \qquad r > 0$$

(geometric region)

Can expand in q^{-1} $(r<0)$.

$$\frac{5q^{-1}}{q^{-1}+5^5} = \text{Series in } q^{-1}$$

coefs are calculated as above

In example 2

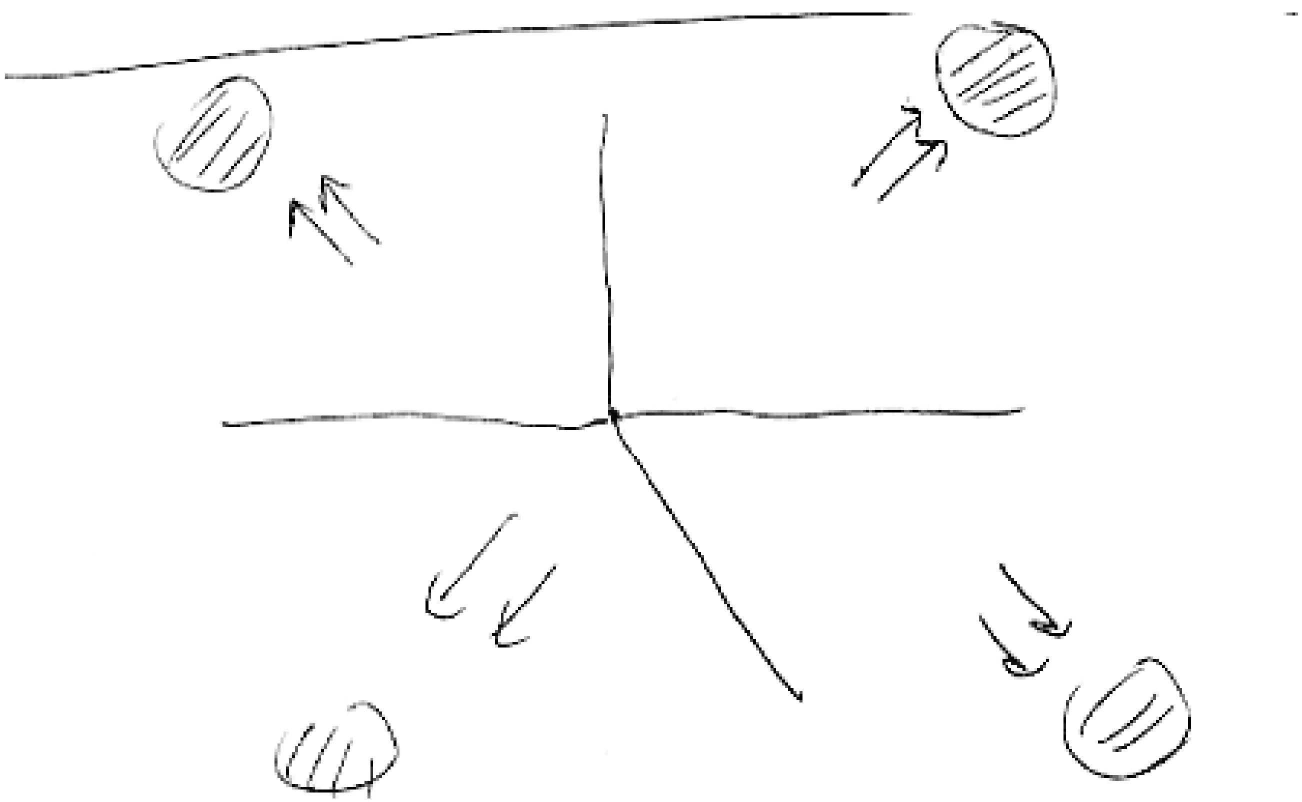
calculations can be made in any region.

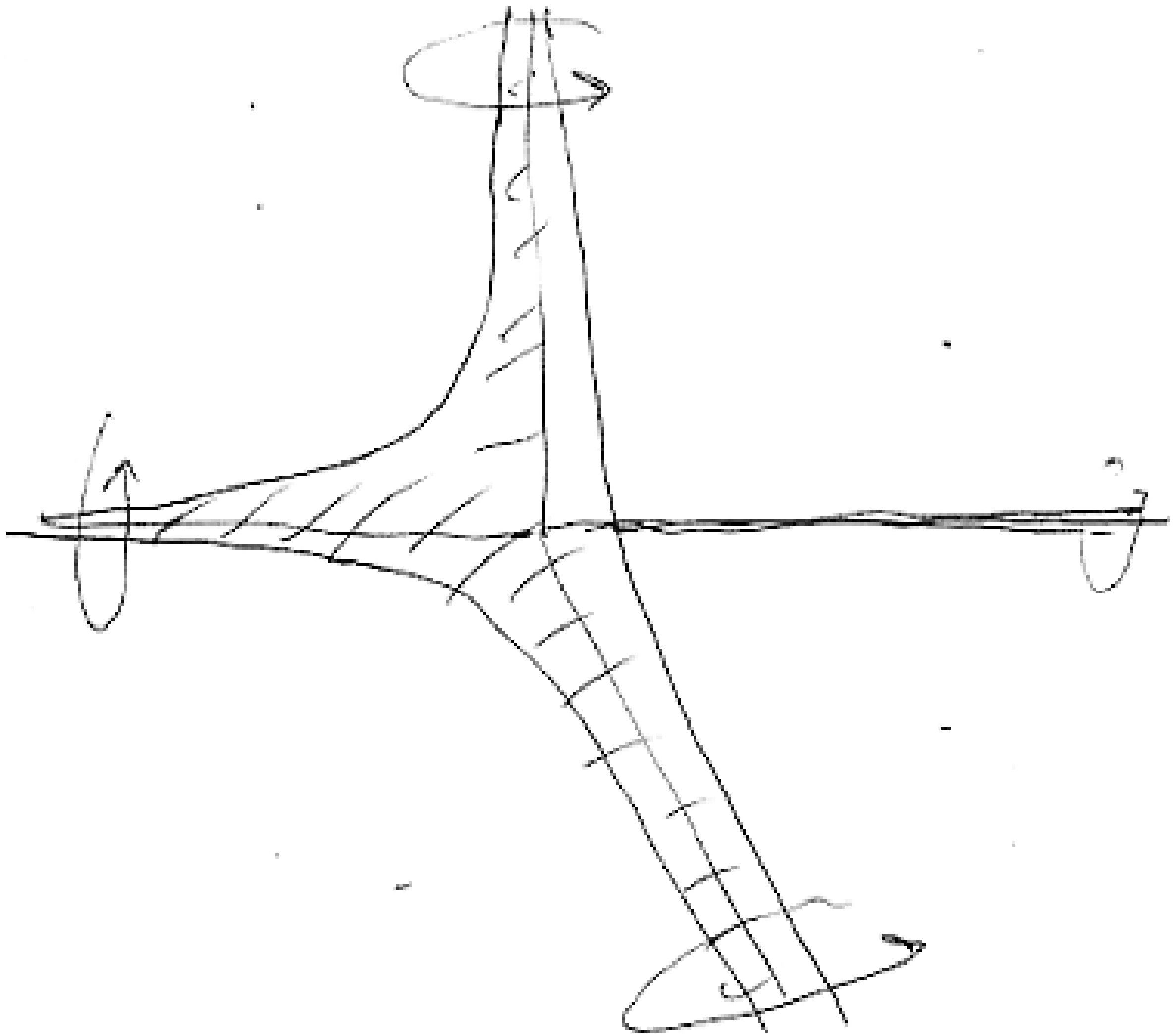
easiest to make in region 3.

$$\chi_j^{\vec{n}} = -2^{8n_1+2n_2+3j}\binom{j-2-2n_2}{j-2-n_1}$$

$\vec{n} \in$ dual cone to region III

weighted with appropriate q's





$$\left\{ \left|\cancel{\Delta}\right|_{(|q_1|,|q_2|)} : \Delta^{(q_1,q_2)} \neq \emptyset \right\}$$

Mysterious aspect: How to analyze the behavior in the ~~...~~ shaded region