# Instanton Sums and Monodromy

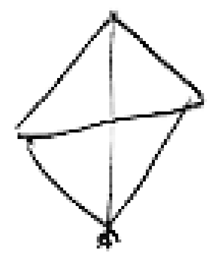
## Lecture 4

$G \longrightarrow \mathbb{C}^n$

$U(1)^{n-d} \longrightarrow U(1)^n$

$\mathbb{Z}^{n-d} \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^d$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \longmapsto v_i \in \mathbb{Z}^d$$



GLSM: convex hull of $v_i$ = Gorenstein cone.

$X_0$

anit $X_0$: reflexive polytope

$$P = S\,T$$

$\Big\downarrow$ "mirror to"

$${}^{t}P = {}^{t}T\;{}^{t}S$$

$\hat{N}$ $\hat{G}$

polar of a reflexive polytope.

— Parameters in mirror symmetry:

$\tau_a \longleftrightarrow$ ~~$\mathfrak{g}$~~ basis of $\mathfrak{g}$.

$q_a = e^{2\pi i \tau_a}$

$\hat{W} = \sum \hat{c}_j x^{m_j}$

$U(1)^d$ acts on $\hat{X}$

form invariant quot under $U(1)^d$ action

$$r_a = -\frac{1}{2\pi} \sum \hat{Q}_i^a \log |\hat{c}_i|$$

$$\frac{1}{2\pi}\theta_a + i r_a = \tau_a$$

$$q_a = e^{2\pi i \tau_a} = \prod |\hat{c}_i|^{Q_i^a} \cdot \text{phase}$$

$$\boxed{q_a = \prod \hat{c}_i^{\hat{Q}_i^a}}$$

## Quintic case

mirror theory.

$$\hat{C}_0 \hat{X}_1 \cdots \hat{X}_5 + \hat{C}_1 \hat{X}_1^5 + \cdots + \hat{C}_5 \hat{X}_5^5 = 0$$

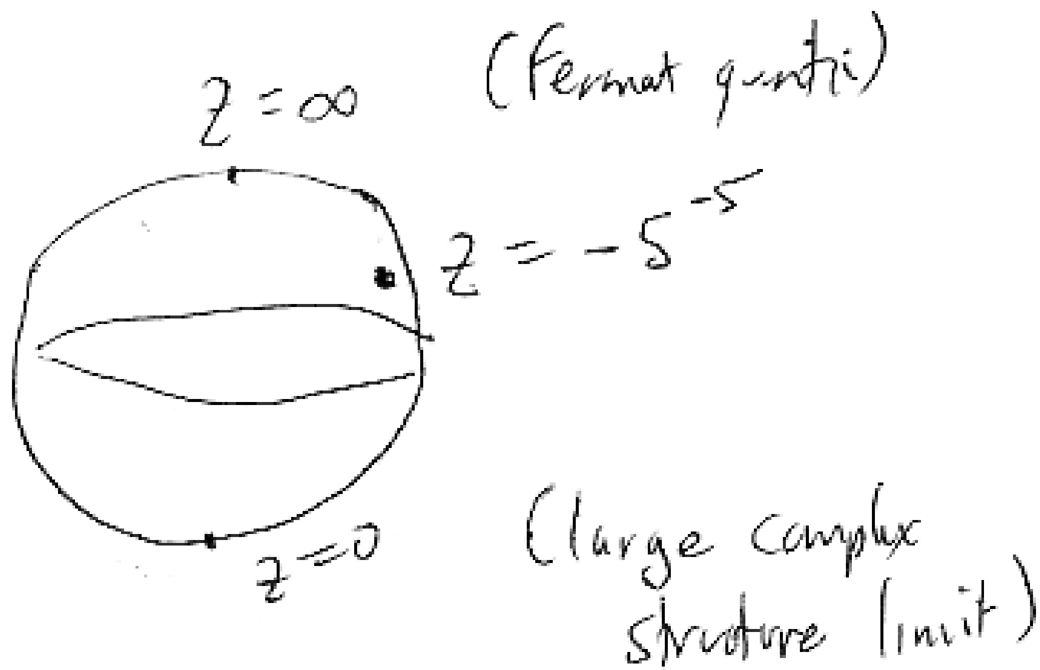
Single invariant quantity

$$z = \frac{\hat{C}_1 \cdots \hat{C}_5}{\hat{C}_0^5}$$

VHS Periods and their monodromy.

period for mirror quintic

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$



$z=\infty$
(Fermat quintic)
$z=-5^{-5}$
$z=0$
(large complex structure limit)

$$\mathcal{D}\Phi = 0.$$

$$\mathcal{D} = \left(z\frac{d}{dz}\right)^4 - 5z\left(5z\frac{d}{dz}+1\right)\left(5z\frac{d}{dz}+2\right)\cdots\left(5z\frac{d}{dz}+4\right)$$

$$\Phi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(5\alpha+1)(5\alpha+2)(5\alpha+3)\cdots(5\alpha+5n)}{[(\alpha+1)(\alpha+2)\cdots(\alpha+n)]^5}\, z^{\alpha+n}$$

$$\mathcal{D}\Phi_\alpha(z) = \alpha^4 z^\alpha.$$

$$\mathbb{C}[\alpha]/(\alpha^4).$$

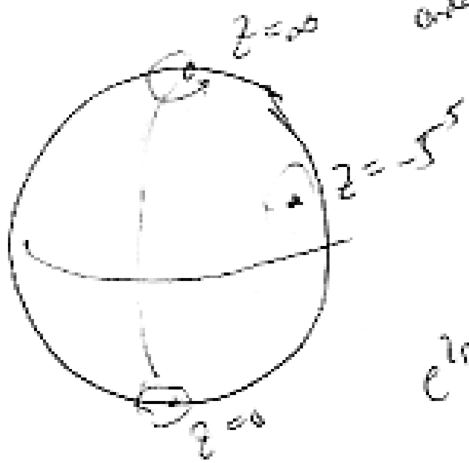
$$\log z \longrightarrow \log z + 2\pi i$$

$$\Phi_\alpha(z) \longmapsto e^{2\pi i\alpha}\,\Phi_\alpha(z)$$

$$\Phi_\alpha(z) = \sum_{m=0}^{\infty} \frac{[\alpha(\alpha+1)\cdots(\alpha+(m-1))]^5}{(5\alpha)(5\alpha+1)\cdots(5\alpha+5m-1)} z^{-\alpha-m}.$$

$$\mathcal{D}\Phi_\alpha(z) = \underbrace{-5(-5\alpha+1)(-5\alpha+2)(-5\alpha+3)(-5\alpha+4)}_{\overset{\shortparallel}{0} \text{ to get solutions}} z^{-\alpha+1}$$

$\times\, e^{2\pi i \alpha}$ = mult. by a $5^{\text{th}}$ root 1.




order 5

(no good formal p.s.)

$e^{2\pi i\alpha}$ on $\mathbb{C}[\lambda]/\lambda^4$

<u>Periods</u> span $H^3(\hat{X}_z)$

natural bundle with fiber $H^{ev}(X)$

near $q=0$:

$(D_1 \cdot D_2 \cdot D_3) \iff$ multiplication on coho, given Poincaré duality pairing

$\langle D_1 \cdot D_2, D_3 \rangle$

$$D_1 \cdot D_2 \cdot D_3 + \sum_{n \geq 0} c_n q^n.$$

is a deformation of ring structure.

<u>Quintic</u> ~~Quantum coh. ring~~

Classical coh ring

$$\mathbb{Q}[\alpha]/(\alpha^4)$$

$e^{2\pi i \alpha}$ in $\mathbb{Q}[\alpha]/(\alpha^4)$

$$T = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ & 1 & 1 & \frac{1}{2} \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

2-dim'l quantum field theory:

string theory has additional features

including "D-branes".

D-branes $\longleftrightarrow$ "bd." Derived category of coherent sheaves on $X$.

Automorphisms of derived category:

$$T = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ & 1 & 1 & \frac{1}{2} \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$

Kontsevich:

$\Delta \subseteq X \times X$, $\mathcal{O}_\Delta$.

$$\mathcal{E} \longmapsto (p_2)_* (p_1^* \mathcal{E} \otimes \mathcal{O}_\Delta).$$

Calabi–Yau is exactly what's required for this to be an automorphism:

$$\gamma \longmapsto \gamma - \left( \int \gamma \wedge \operatorname{Todd} T_X \right) 1_X .$$

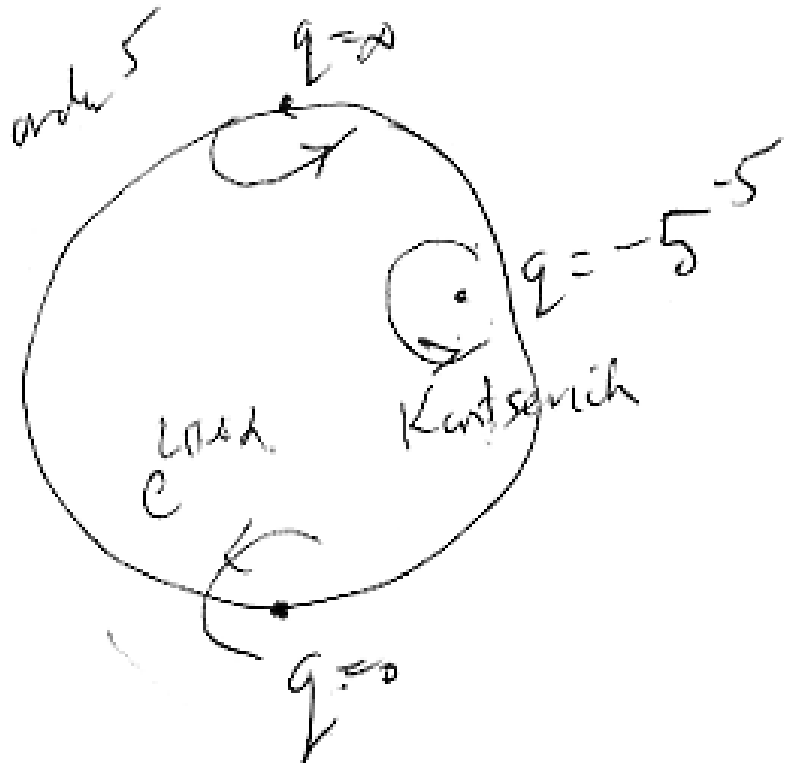
$1_X$ generates $H^0(X)$.

Calculate for quintic:

$$S = \begin{pmatrix} 1 & & & \\ -\frac{25}{6} & 1 & & \\ 0 & 0 & 1 & \\ -5 & 0 & 0 & 1 \end{pmatrix}$$

$$TS = \begin{pmatrix} -4 & 1 & \frac{1}{2} & \frac{1}{6} \\ -\frac{20}{3} & 1 & 1 & \frac{1}{2} \\ -5 & 0 & 1 & 1 \\ -5 & 0 & 0 & 1 \end{pmatrix}$$

$(TS)^5 = \text{identity}$


order 5
q=∞
q = −5^{-5}
Konts...ich
cond.
C
q=0

Kontsevich verified this proposal for all known 1-param examples with a Fermat point.

There is a bundle over moduli, represent D-branes in physics, and there should geometric transformations corresponding to monodromy around loops.

Start with a sheaf $\mathcal{F}$.

$$d_{\mathcal{F}} = \mathrm{Ker}\left(\mathcal{F} \boxtimes \mathcal{F}^* \to \mathcal{O}_{\Delta}\right)$$

$d_{\mathcal{F}}$ is on $X \times X$.

$$\mathcal{E} \mapsto (p_2)_*\left(p_1^*\mathcal{E} \otimes d_{\mathcal{F}}\right).$$

$$\mathrm{Ext}^k(\mathcal{F}, \mathcal{F}) = \begin{cases} \mathbb{C} & \text{if } k = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

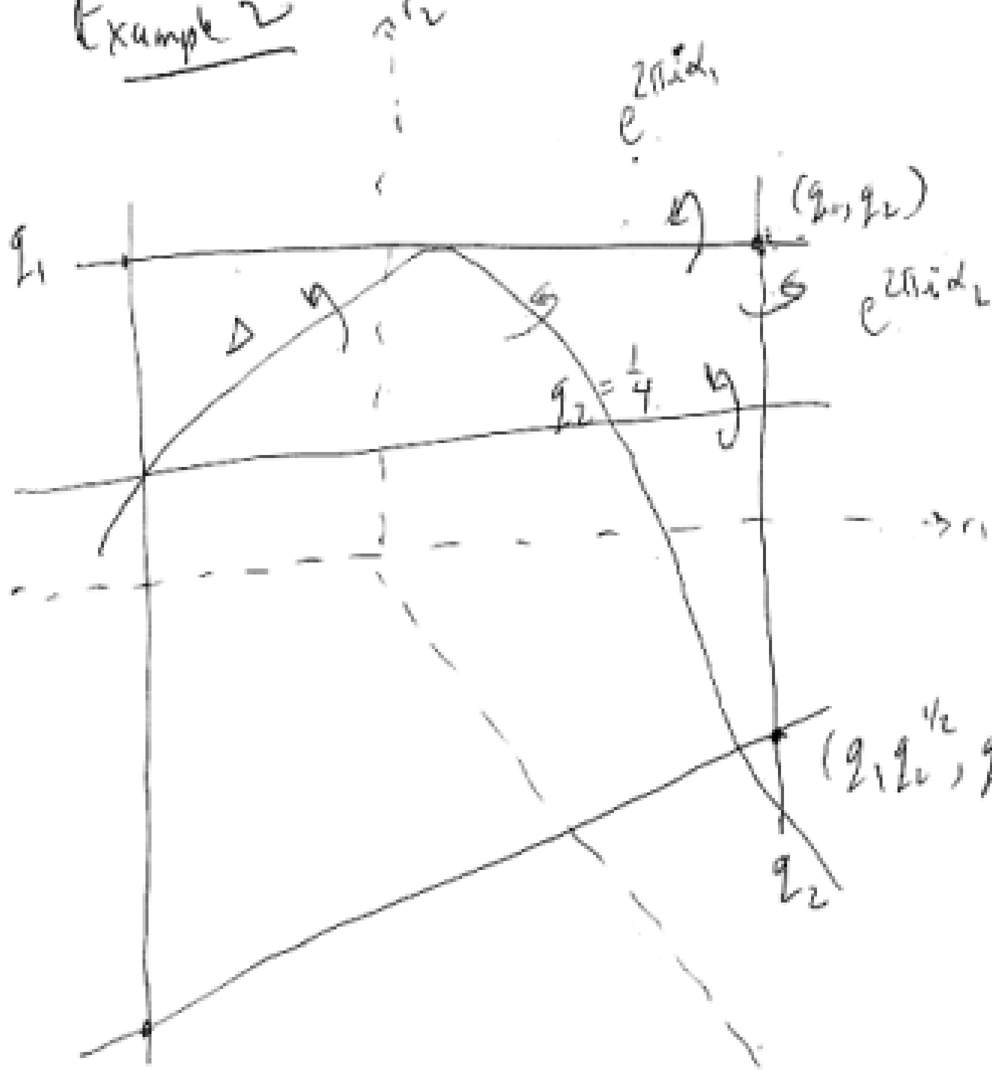
"homology sphere for ext"

$\mathcal{F} = \mathcal{O}_X$. $\quad \mathrm{Ext}^k(\mathcal{O},\mathcal{O}) = H^k(\mathcal{O}_X)$

Seidel + Thomas, Horja:

many examples have been checked.

# Example 2


$r_2$
$e^{2\pi i \alpha_1}$
$(q_1, q_2)$
$q_1$
$e^{2\pi i \alpha_2}$
$\Delta$
$q_2 = \frac{1}{4}$
$r_1$
$(q_1 q_2^{1/2}, q_2^{-1/2})$
$q_2$