Instanton Sums and Monodromy
Lecture 2

$\rho : G \to U(1)^n$, acts on $\mathbb{C}^n$ G-invariant $W(x_1, \ldots, x_n)$

G = compact abelian

$\mu : \mathbb{C}^n \longrightarrow \mathfrak{g}^*$

$\mu^{-1}(r)/G =$ a Kähler space
(sometimes a manifold)

top. depends on location of $r \in \mathrm{Im}(\mu)$.

$X = \mu^{-1}(r)/G$ is a toric variety.

Coordinate charts: Say $\operatorname{rank} G = n-d$.
$\Rightarrow \dim_{\mathbb{C}} X = d$.

Consider subsets of d of the homog. coords

$$X_{i_1}, \ldots, X_{i_d}.$$

Does open set in $\mathbb{C}^d$ containing $\vec{0}$ descend to a subset of X?

<u>Example</u>

$$\begin{array}{cccc} X_0 & X_1 & \cdots & X_5 \\ -5 & 1 & \cdots & 1 \end{array}$$

$G = U(1)$

$$\mu = -5|x_0|^2 + |x_1|^2 + \cdots + |x_5|^2$$

$x_0\, x_1, \ldots, x_4$ a coord chart?
μ -

$$|x_5|^2 = r + 5|x_0|^2 - |x_1|^2 - \cdots - |x_4|^2$$

if $r > 0$, open set contain $\vec{0}$
on which can be solved.

Use $U(1)$ to make x_5 real.

In the example,

$$\mu^{-1}(r)/G = \text{total space of } \mathcal{O}(-5) \text{ over } \mathbb{CP}^4$$

$$\frac{\partial W}{\partial \phi_i} = 0.$$

$$W = x_0(x_1^5 + \cdots + x_5^5 + x_1 x_2 x_3 x_4 x_5)$$

$$\Rightarrow \quad x_0 = 0, \quad x_1^5 + \cdots + x_5^5 + x_1 \cdots x_5 = 0.$$

On a quintic hypersurface in $\mathbb{CP}^4$.

Families of CY manifolds:

2 types of params: coefs in W.

Kähler classes on X (complexified)

τ.

$\prod x_i^{p_{ir}}$ $(p_{ir}) = P$ nonnegative integers

Laurent basis for F inv. $\prod x_i^{t_{ia}}$ $(t_{ia}) = T$ integer matrix.

$P = S T$

$$\prod x_i^{p_{ir}} = \prod \left(\prod x_i^{t_{ia}} \right)^{s_{ar}}$$

In example: $x_0 x_1^5, \ldots, x_0 x_5^5, x_0 x_1 \cdots x_5$

$$P = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Laurent basis: $x_0 x_1^5$, x_i/x_0.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{= S} \underbrace{\begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{= T}$$

Mirror:

replace P, S, T by

$$t^P,\ t^T,\ t^S$$

$$t^P = t^{T^{t^S}}$$

In examples for mirror group is

$$U(1) \times \mathbb{Z}_5^3$$

(Briefly skip analysis of 1-loop effects)

Instanton expansions:

Cohomology of toric variety
(applies if toric variety is compact)

~~case~~.

For $x_{i_1}, \ldots x_{i_d}$ to be coords in some chart, need $\rho|_{\mathrm{span}(x_{i_1}, \ldots x_{i_d})}$ has at worst a finite kernel.

if there is a finite kernel,

$\mathbb{C}^d / \mathrm{Ker.}$, singularities.

$H^*(X, \mathbb{Q})$ can still be calculated.

<u>Fact</u>: $H^*(X, \mathbb{Q})$ is generated as $\mathbb{Q}$-algebra by $H^2(X)$.

$H^2(X)$ gen. by $c_1(\{x_i = 0\}) =: \xi_i$

$G \to U(1)^n$. $X = \mu^{-1}(r)/G$.

any character of G. determine a $U(1)$-bundle (or $\mathbb{C}^*$-bundle).

can identify $H^2(X,\mathbb{Q}) = \mathrm{Char}(G) \otimes \mathbb{Q}$.

Specifically,

if G acts on x_i by χ_i,

$$\chi_i = \prod (e^{\eta_a})^{Q_a^i}$$

exponents describe group action.

e^{η_a} = basis of $\mathrm{char}(G) \otimes \mathbb{Q}$.

η_a basis for $H^2(X, \mathbb{Q})$

$$\boxed{\xi_i = \sum Q_a^i \eta_a}$$

$$\xi_{i_1} \cdot \ldots \cdot \xi_{i_d} = \text{intersection number.}$$

$$\# (X_{i_1}=0) \cap \ldots \cap (X_{i_d}=0) = \begin{cases} 0 & \text{if } \underline{\text{not}} \text{ a coord chart} \\ \dfrac{1}{\#(\operatorname{Ker} \rho)} \end{cases}$$

Relations in ring

1) linear relations

$$\xi_i = \sum Q^i_a \eta_a$$

2) nonlinear relation

$$\xi_{i_1} \cdot \ldots \cdot \xi_{i_k} = 0$$

if $(x_{i_1}=0) \cap \ldots \cap (x_{i_k}=0) = \emptyset$

on space.

(Stanley-Reisner ideal)

Thm this is a complete set of relations.

$$\left[\xi_{i_1}\cdot\ldots\cdot\xi_{i_d}\right]\Big|_{[X]} = ?$$

Correlation functions in (topological) GLSM.

$$\Sigma \longrightarrow \mathbb{C}^n$$

governed by $\mathcal{L}$.

$\mu^{-1}(r)/G$.

$$\frac{\partial W}{\partial \phi_i} = 0.$$

φ_i = low components of chiral fields.

$\Phi : M^{2|(2,2)} \to \mathbb{C}^N$

$\Phi|_{M^2} = \varphi : M^2 \to \mathbb{C}^N$

$\langle\langle \sigma_1 \cdots \sigma_k \rangle\rangle$

σ_j = a certain point P_j on Σ

should be mapped to a divisor

$\xi_j := \{x_j = 0\}$.

classical term:

insist that map $\Sigma \to \mathbb{C}^n$ be

top. trivial, and minimize energy.

$\Rightarrow$ map is constant.

$\Sigma = \mathbb{CP}^1$.

$\mathbb{CP}^1 = \mathbb{C}^2 / \mathbb{C}^*$

$\mu_\Sigma : \mathbb{C}^2 \to \mathbb{R}$ for $U(1)$

$\mathbb{CP}^1 = \mu_\Sigma^{-1}(r)/U(1)$.

$\mathbb{C}^2 \longrightarrow \mathbb{C}^n$. equivariant.

$\lambda : U(1) \longrightarrow G$

λ specifies the homology class of image $\mathbb{CP}^1$.

$U(1) \xrightarrow{\lambda} G \xrightarrow{\chi} U(1)$ $\deg(\chi \circ \lambda) = \# D_\chi \cdot \Sigma$.

$\mathbb{C}^2 \longrightarrow \mathbb{C}^n$? $\lambda : U(1) \to G$

$x_j = f_j(s,t)$ hom. of deg d_j

when $d_j = \deg(\cancel{\lambda \circ \chi_j})$ $(\chi_j \circ \lambda)$

top. trivial: $d_j = 0 \quad \forall_\sigma$:

x_j = constant on $\mathbb{C}^2$.

σ_k = restricts us to maps f s.t.

$$f(P_k) \in (X_k = 0)$$

if f is constant, means the image point lies on $(X_k = 0)$.

$$\langle\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle\rangle_{\text{class}} = \#(x_{i_1} = 0) \cap \cdots \cap (x_{i_k} = 0).$$

$\langle\langle \sigma_{i_1} \cdots \sigma_{i_d} \rangle\rangle_{\text{class}}$ can be calculated as $\frac{1}{\#(\ker)}$ or 0.

$$\frac{\partial W}{\partial x_i} = 0.$$

$-5 \quad 1 \cdots 1$

Restrict to $W = x_0 P(x_1, \ldots, x_n)$

$d_j \geq 0$ if $j \geq 1$

$d_0 < 0$

poly of neg. degree is $\equiv 0$.

$\boxed{x_0 \equiv 0.}$

$$P(f_1(s,t), \ldots, f_n(s,t)) \equiv 0$$

poly. of degree $-d_0 > 0$.

This will vanish identically if it vanishes

at $-d_0+1$ points.

divisor class of P = anticanonical class of X.

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \Big|_{\square}$$

$$\langle\langle \sigma_{i_1} \cdots \sigma_{i_k} \rangle\rangle_{GLSM}$$

$$\shortparallel$$

$$\sum_{\vec{d}} (-1)^{d_0} \langle \sigma_{i_1} \cdots \sigma_{i_k} \, (-K)^{-d_0+1} \, \blacksquare \rangle_{\vec{d}} \, q^{\vec{d}}$$

$\vec{d} = \vec{0}$
is classical.

$$q = e^{2\pi i \tau}.$$

# of maps $\mathbb{C}^L \to \mathbb{C}^n$ of degrees d_j.

$\sigma_{i_1}(P_{i_1}) \in \{x_{i_1}=0\}$

also conditions for $P(x_1,\dots,x_n)=0$.

Quintic

$H^2(X) = \mathbb{Z}$.

$\eta \ldots$

$$\langle\langle \eta\,\eta\,\eta \rangle\rangle_{GLSM} = \sum c_n q^m$$

$$c_n = (-1)^{d_0} \langle \eta\,\eta\,\eta\,(-K)^{-d_0+1} \rangle_m$$

$$-d_0 = 5m$$

$x_j = f_m(s,t) \quad j = 1 \ldots n$

x_0 has dge $-5m$.

$P(\quad)$ has degree $5m$

$\# K = -5\eta$.

$$-\langle \eta^3 (-5\eta)^{5m+1} \rangle_m = -\langle \eta^{5m+4} (-5)^{5} \ldots$$

$$= -(-5)^{5m+1}$$